Unbounded operators in Hilbert spaces and EPDEs with boundary conditions

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August 29, 2022

1 Basic Settings

Let $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_{\mathcal{H}_1})$,$(\mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_2})$be Hilbert spacces. We say $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$, $\mathrm{T} : \mathcal{H}_1 \mapsto \mathcal{H}_2$ is unbounded linear operator, if restricted in a dense subspace $\mathcal{D}(\mathrm{T}) \subset \mathcal{H}_1$, T is linear. Moreover, we say $\mathcal{D}(\mathrm{T})$ is the domain of T.

Example 1. *Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R})$, $\mathrm{T} = \frac{d}{dx}$, $\mathcal{D}(\mathrm{T}) = C_c^\infty(\mathbb{R})$. Then $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ is an unbounded operator.*

Let $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1))$ and $(\mathrm{T}_2, \mathcal{D}(\mathrm{T}_2))$ $(T_i : \mathcal{H}_1 \mapsto \mathcal{H}_2)$ be unbounded operators, if $\mathcal{D}(\mathrm{T}_1) \subset \mathcal{D}(\mathrm{T}_2)$ and $\mathrm{T}_2 |_{\mathcal{D}(\mathrm{T}_1)} = \mathrm{T}_1$, we say $(\mathrm{T}_2, \mathcal{D}(\mathrm{T}_2))$ is an extension of $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1))$, denoted by $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1)) < (\mathrm{T}_2, \mathcal{D}(\mathrm{T}_2))$.

Remark 1. *If there exists $M > 0$, such that $\forall x \in \mathcal{D}(\mathrm{T})$, $\| \mathrm{T} x\| \leq M\|x\|$, Then T could be extended to a linear operator, with domain $\mathcal{H}_1$.*

Next, we always assume $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ is closable: If $\{x_n\}_{n=1}^\infty \subset \mathcal{D}(\mathrm{T})$, such that $\lim_{n\to\infty} x_n = 0$, and $\lim_{n\to\infty} \mathrm{T} x_n$ exists, then we must have $\lim_{n\to\infty} \mathrm{T} x_n = 0$.

Remark 2.

1. *The unbounded operator in Example 1 is closable: let $f_0 \in C_c^\infty(\mathbb{R}) \to 0$ in $L^2(\mathbb{R})$, and $f_n' \to g$ for some $g \in L^2(\mathbb{R})$. If $g \neq 0 \in L^2(\mathbb{R})$, since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists $h \in C_c^\infty(\mathbb{R})$, s.t. $\langle g, h\rangle_{L^2(\mathbb{R})} \neq 0$. But*
$$\langle g, h\rangle_{L^2(\mathbb{R})} = \lim_{n\to\infty} \langle f_n', h\rangle_{L^2(\mathbb{R})} = -\lim_{n\to\infty} \langle f_n, h'\rangle_{L^2(\mathbb{R})} = 0.$$
As a result, we must have $g = 0$.

2. *Let $\mathcal{H}_1 = L^2(\mathbb{R}), \mathcal{H}_2 = \mathbb{R}$, $\mathcal{D} = C_c(\mathbb{R}) \subset \mathcal{H}$. Consider the unbounded operator $(\mathrm{T}, \mathcal{D})$, $f \to f(0)$. Then $(\mathrm{T}, \mathcal{D})$ is not closable: Let*
$$f_n(x) = \begin{cases} n(x + 1/n), & \text{if } x \in (-1/n, 0) \\ n(1/n - x), & \text{if } x \in (0, 1/n) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$
Then $f_n \in C_c(\mathbb{R})$ and $f_n \to 0$ in $L^2(\mathbb{R})$. Moreover, $f_n(0) = 1$, hence $\lim_{n\to\infty} Tf_n = 1 \neq 0$, which means T is not closable.

Definition 1. *We say that* $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ *is a close operator, if for a Cauchy Sequence* $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}_1$*such that* $\{\mathrm{T}\, x_n\} \subset \mathcal{H}_2$ *is also a Cauchy sequence, then* $x := \lim_{n\to\infty} x_n \in \mathcal{D}(\mathrm{T})$*, and* $\mathrm{T}\, x = \lim_{n\to\infty} \mathrm{T}\, x_n$.

Definition 2 (close extension)**.** *We say* $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1))$ *is a close extension of* $(\mathrm{T}_0, \mathcal{D}(\mathrm{T}_0))$*, if*

1. $(\mathrm{T}_1, \mathcal{D}_1)$ *is closed;*
2. $\mathcal{D}(\mathrm{T}_0) \subset \mathcal{D}(\mathrm{T}_1)$*;*
3. $\mathrm{T}_1 |_{\mathcal{D}(\mathrm{T}_0)} = \mathrm{T}_0$.

Let $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ be an unbounded operator. For $x, y \in \mathcal{D}(\mathrm{T})$, define the inner product $\langle \cdot, \cdot \rangle_{\mathrm{T}}$:

$$\langle x, y \rangle_{\mathrm{T}} := \langle x, y \rangle_{\mathcal{H}_1} + \langle \mathrm{T}\, x, \mathrm{T}\, y \rangle_{\mathcal{H}_2}.$$

It's easy to check that if $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ is closed, then $\mathcal{D}(\mathrm{T})$ is complete with respect to the norm $\|\cdot\|_{\mathrm{T}}$.

Let $\mathcal{D}(\bar{\mathrm{T}}_{min})$ be the completion of $\mathcal{D}(\mathrm{T})$ under the norm $\|\cdot\|_{\mathrm{T}}$. Since $\|x\|_{\mathcal{H}_1} \leq \|x\|_{\mathrm{T}}, \forall x \in \mathcal{D}(\mathrm{T})$, we can think $\mathcal{D}(\bar{\mathrm{T}}_{min})$ as a dense subspace of $\mathcal{H}_1$. $\forall x \in \mathcal{D}(\bar{\mathrm{T}}_{min})$, since T is closable, define $\bar{\mathrm{T}}_{min} x = \lim_{n\to\infty} \mathrm{T}\, x_n$, where $\lim_{n\to\infty} \|x_n - x\|_{\mathrm{T}} = 0$. Then one can show that $(\bar{\mathrm{T}}_{min}, \mathcal{D}(\bar{\mathrm{T}}_{min}))$ is a close extension of $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$, called minimal extension of $(\mathrm{T}, \mathcal{D}(T))$. Moreover, if $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1))$ is another close extension of $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$, then $(\bar{\mathrm{T}}_{min}, \mathcal{D}(\bar{\mathrm{T}}_{min})) < (\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1))$.

Example 2.
1. *Let* Ω *be a bouned domain in* $\mathbb{R}^n$ *with smooth boundary. Let* $\mathcal{H}_1 = L^2(\Omega)$, $\mathcal{H}_2 = \underbrace{L^2(\Omega) \oplus ... \oplus L^2(\Omega)}_{n \text{ copies of } L^2(\Omega)}$, $\mathcal{D} = C_c^{\infty}(\Omega)$*. Define* $\mathrm{T} : \mathcal{H}_1 \mapsto \mathcal{H}_2$*:*

$$\phi \to (\frac{\partial}{\partial x_1}\phi, ..., \frac{\partial}{\partial x_n}\phi), \forall \phi \in C_c^{\infty}.$$

Then $\mathcal{D}(\bar{\mathrm{T}}_{min})$ *is the Sobolev space* $W_0^{1,2}(\Omega)$*,* T *is the weak derivatives (See page 245 in [1] for more details).*

2. *Now let*

$$\mathcal{D} = \{\phi \in C^{\infty}(\Omega) : \ \phi \text{ and } \partial_{x_i}\phi \text{ are } L^2\text{-integable}\}.$$

Then $\mathcal{D}(\bar{\mathrm{T}}_{min})$ *is the Sobolev space* $W^{1,2}(\Omega)$ *(See Theorem 2 in page 251 of [1]).*

2 Adjoint operator

Definition 3 (Formal adjoint operator)**.** *We say* $(\mathrm{S}, \mathcal{D}(\mathrm{S}))$ *is a formal adjoint operator of* $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$*, if* $\forall x \in \mathcal{D}(\mathrm{T}), y \in \mathcal{D}(\mathrm{S})$*,*

$$\langle \mathrm{T}\, x, y \rangle_{\mathcal{H}_2} = \langle x, \mathrm{S}\, y \rangle_{\mathcal{H}_1}.$$

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ *and* $(\mathrm{S}, \mathcal{D}(\mathrm{S})) = (\mathrm{T}, \mathcal{D}(\mathrm{T}))$*, then we say* $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ *is symmetric.*

It could be check easily that if $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ has a formal adjoint operator $(\mathrm{S}, \mathcal{D}(\mathrm{S}))$, then $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ is closable: let $x_n \in \mathcal{D}(\mathrm{T})$, such that $x_n \to 0$, $\mathrm{T}\, x_n \to g$. If $g \neq 0$, one can find $h \in D(\mathrm{S})$, such that $(g, h) \neq 0$. But

$$\langle g, h \rangle_{\mathcal{H}_2} = \lim_{n\to\infty} \langle T x_n, h \rangle_{\mathcal{H}_2} = \lim_{n\to\infty} \langle x_n, S h \rangle_{\mathcal{H}_1} = 0,$$

which is a contradiction.

In fact, if $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ is closable, then it has a special formal adjoint operator, called adjoint operator:

Definition 4 (Adjoint Operator)**.** *We say that* $(\mathrm{T}^*, \mathcal{D}(\mathrm{T}^*))$ *is the adjont operator of* $(\mathrm{T}, D(\mathrm{T}))$*, if* $(\mathrm{T}^*, \mathcal{D}(\mathrm{T}^*))$ *is a formal adjoint operator of* $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$*, and*

$$\mathcal{D}(\mathrm{T}^*) := \{y \in \mathcal{H}_2 : \text{ there exists } M_y > 0 \text{such that } |\langle \mathrm{T}\, x, y\rangle_{\mathcal{H}_2}| \leq M_y \|x\|_{\mathcal{H}_1}, \forall x \in \mathcal{D}(\mathrm{T})\}.$$

If $\mathcal{H}_1 = \mathcal{H}_2$*,* $(\mathrm{T}^*, \mathcal{D}(\mathrm{T}^*)) = (\mathrm{T}, \mathcal{D}(\mathrm{T}))$*, then we say* $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ *is self-adjoint.*

It's easy to check that if $(\mathrm{T}_1, \mathcal{D}(\mathrm{T}_1)) < (\mathrm{T}_2, \mathcal{D}(\mathrm{T}_2))$, then

$$(\mathrm{T}_2^*, \mathcal{D}(\mathrm{T}_2^*)) < (\mathrm{T}_1^*, \mathcal{D}(\mathrm{T}_1^*)).$$

Moreover, it follows from the definition that $(\mathrm{T}^*, \mathcal{D}(\mathrm{T}^*))$ is closed: let $\{y_n\} \subset \mathcal{D}(\mathrm{T}^*)$ be a Cauchy sequence, s.t. $\mathrm{T}^*(y_n)$ is a Cauchy sequence in $\mathcal{H}_1$. Let $y = \lim_{n\to\infty} y_n \in \mathcal{H}_2$, $z = \lim_{n\to\infty} \mathrm{T}^*(y_n) \in \mathcal{H}_1$, then for all $x \in D(\mathrm{T})$,

$$|\langle \mathrm{T}\, x, y\rangle_{\mathcal{H}_2}| = \lim_{n\to\infty} |\langle \mathrm{T}\, x, y_n\rangle_{\mathcal{H}_2}| = \lim_{n\to\infty} |\langle x, \mathrm{T}^*\, y_n\rangle_{\mathcal{H}_1}| = |\langle x, z\rangle_{\mathcal{H}_1}| \leq \|z\|_{\mathcal{H}_1} \|x\|_{\mathcal{H}_1}.$$

Hence, one can see that $y \in \mathcal{D}(\mathrm{T}^*)$, moreover $T^* y = z$.

In fact, one has $(\mathrm{T}^{**}, \mathcal{D}(\mathrm{T}^{**})) = (\bar{\mathrm{T}}_{min}, \mathcal{D}(\bar{\mathrm{T}}_{min}))$.

If $(\mathrm{S}, \mathcal{D}(\mathrm{S}))$ is a formal adjoint operator of $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ then $(\mathrm{S}^*, \mathcal{D}(\mathrm{S}^*))$ is a close extension of $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$.

Example 3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary. Let* $\mathcal{H}_1 = L^2(\Omega)$*,* $\mathcal{H}_2 = \underbrace{L^2(\Omega) \oplus ... \oplus L^2(\Omega)}_{n \text{ copies of } L^2(\Omega)}$*,* $\mathcal{D} = C_c^\infty(\Omega)$*. Set* $\mathrm{T} : \mathcal{H}_1 \mapsto \mathcal{H}_2$*:*

$$\phi \to (\frac{\partial}{\partial x_1}\phi, ..., \frac{\partial}{\partial x_n}\phi), \forall \phi \in C_c^\infty.$$

Set $\mathcal{D}^n := \underbrace{C_c^\infty(\Omega) \oplus ... \oplus C_c^\infty(\Omega)}_{n \text{ copies of } C_c^\infty(\Omega)}$*, and* $\mathrm{S} : \mathcal{H}_2 \mapsto \mathcal{H}_1$*,*

$$(\phi_1, ...\phi_n) \to -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\phi_k, \phi_k \in C_c^\infty(\Omega),$$

Then $(\mathrm{S}, \mathcal{D}^n)$ *is a formal adjoint operator of* $(\mathrm{T}, \mathcal{D})$*. Moreover, it follows from the definition of Sobolev space that* $\mathcal{D}(\mathrm{S}^*) = W^{1,2}(\Omega)$*. Here we give another desciption of Sobolev space* $W^{1,2}(\Omega)$*.*

3 Friedrichs Extension and Essential self-adjoint

Let $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$ be a nonnegative symmetric operator, that is, for all $\phi \in \mathcal{D}(\mathrm{T})$,

$$\langle T\phi, \phi\rangle_{\mathcal{H}} = \langle \phi, T\phi\rangle_{\mathcal{H}} \geq 0.$$

Then, on $\mathcal{D}(\mathrm{T})$,

$$\langle \phi, \psi \rangle_{\mathrm{T}^{1/2}} := \langle \phi, \psi \rangle_{\mathcal{H}} + \langle \phi, T\psi \rangle_{\mathcal{H}}, \phi, \psi \in \mathcal{H}$$

defines an inner product. Let $\mathcal{H}_1$ be the compection of $\mathcal{D}(\mathrm{T})$ under the norm $\|\cdot\|_{T^{1/2}}$ then $\mathcal{H}_1$ could be think as a subspace of $\mathcal{H}$. Set

$$\mathcal{D}^F := \{\phi \in \mathcal{H}_1 : \langle \eta, \phi \rangle_H + \langle T\eta, \phi \rangle_{\mathcal{H}} \le M_\phi \|\eta\|_{\mathcal{H}} (\forall \eta \in \mathcal{D}(\mathrm{T})) \text{ for some } M_\phi > 0.\}$$

By Riesz representation theorem, there exists $u \in \mathcal{H}$, such that

$$\langle \eta, \phi \rangle_H + \langle T\eta, \phi \rangle_{\mathcal{H}} = \langle \eta, u \rangle_{\mathcal{H}}. \tag{2}$$

Now set $\mathrm{T}^F(\phi) = u - \phi$. We called $(\mathrm{T}^F, \mathcal{D}^F)$ be Friedrichs extension of $(\mathrm{T}, \mathcal{D}(T))$. One can check that $(\mathrm{T}^F, \mathcal{D}^F)$ is a closed extension of $(\mathrm{T}, \mathcal{D}(\mathrm{T}))$, and is self-adjoint.

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R}^n)$, $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$, then the operator $\mathrm{T} = \Delta$, $\phi \to \Delta\phi := -\sum_i \partial_i^2 \phi$ is symmetric. Then, $u \in \mathcal{D}^F$ iff $u \in W_0^{1,2}(\Omega)$ solve EPDEs below weakly for some $g \in L^2(\mathbb{R}^n)$:*

$$\begin{cases} \Delta u = g, \ \ in\ \Omega; \\ u = 0, \ \ on\ \partial\Omega, \end{cases} \tag{3}$$

i.e., for all $v \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega hv.$$

Futhermore, $\mathrm{T}^F u = g$.

Next, let $\mathcal{D}^N = \{u \in C^\infty(\bar{\Omega}) : \partial_\nu u = 0 \ on\ \Omega\}$ be the domain of $\mathrm{T}^N = \Delta$, then $u \in (\mathcal{D}^N)^F$ iff $u \in W^{1,2}(\Omega)$ solves EPDEs below weakly for some $h \in L^2(\mathbb{R}^n)$:

$$\begin{cases} \Delta u = h, \ \ in\ \Omega; \\ \partial_\nu u = 0, \ \ on\ \partial\Omega, \end{cases} \tag{4}$$

i.e., for all $v \in W^{1,2}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega hv.$$

Furthermore, $(T^F)^* u = g$.

Here ν is the normal direction on $\partial\Omega$.

Proof. If $u \in \mathcal{D}^F$, then there exists $g \in L^2(\Omega)$, s.t. for any $\eta \in C_c^\infty(\Omega)$

$$\langle \Delta\eta, u \rangle_{L^2(\Omega)} = \langle \eta, g - u \rangle_{L^2(\Omega)}.$$

While integration by parts shows that $\langle T\eta, u \rangle_{L^2(\Omega)} = \int_\Omega \nabla\eta \cdot \nabla u = \int_\Omega \eta(g-u)$. Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, one can see that u solves

$$\begin{cases} \Delta u = g - u, \ \text{in}\ \Omega \\ u = 0, \ \text{on}\ \partial\Omega. \end{cases} \tag{5}$$

On the other hand, if $u \in W_0^{1,2}(\Omega)$ solves (3) for some g, integration by parts shows that

$$\langle \Delta\eta, u\rangle_{L^2(\Omega)} + \langle \eta, u\rangle_{L^2(\Omega)} \leq (\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})\|\eta\|_{L^2(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$. Hence $u \in \mathcal{D}^F$

For Neumann's case, the proof is similar. The only somewhat nontrivial part is to show that $\mathcal{D}^N$ is dense in $W^{1,2}(\Omega)$ (w.r.t. to the norm $\|\cdot\|_{W^{1,2}(\Omega)}(\Omega)$): First, since $C^\infty(\bar\Omega)$ is dense in $W^{1,2}(\Omega)$, for $u \in \mathcal{D}^N$, any $\epsilon > 0$, there exists $v \in C^\infty(\bar\Omega)$, s.t. $\|u - v\|_{W^{1,2}(\Omega)} < \epsilon/2$. Fix $\eta \in C_c^\infty(\mathbb{R})$, s.t. $\mathrm{supp}\eta \supset (-1,1)$, $\eta|_{(-1/2,1/2)} \equiv 1$. Set $M = \int_{\partial\Omega} |\partial_\nu v|^2 + \int_{\partial\Omega} |\nabla^{\partial\Omega}\partial_\nu v|^2$. Let $d(x) := dist(x,\Omega)$, $w(x) = d(x)\eta(NMd(x))\partial_\nu v$, then when $N > 0$ is big, $\|w\|_{W^{1,2}(\Omega)} \leq \frac{C}{N}$ for some $C > 0$ depending only on Ω. Furthermore, $\partial_\nu w = \partial_\nu v$. Then for N is big enough, $\|u-(v+w)\| \leq \epsilon$, and $\partial_\nu(v+w) = 0$. Hence, $\mathcal{D}^N$ is dense in $W^{1,2}(\Omega)$.

□

Moreover, one has

Theorem 1. *When $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, then T^F (or $(\mathrm{T}^N)^F$) has discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \cdots$ (or respectively, $0 \leq \lambda_{1,N} \leq \lambda_{2,N} \leq \cdots \leq \lambda_{k,N} \cdots$). Moreover, their eigenfunctions $\{e_k\}$ (or $\{e_{k,N}\}$) respectively) forms an orthonormal basis of $L^2(\Omega)$. Furthermore, $\lim_{k\to\infty} \lambda_k = \infty$ (or $\lim_{k\to\infty} \lambda_{k,N} = \infty$ respectively).*

4 min-max principle and EPDEs with boundary conditions

In this section, we would like to present another description of eigenvalues of Laplacian operator. For any vector space L, let $\Phi_k(L)$ denote the set of k-dimensional vector spaces.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary.

For $u \in W_0^{1,2}(\Omega)$ or $u \in W^{1,2}(\Omega)$, consider the fuctional

$$\mathcal{F}(u) = \frac{\int_\Omega |\nabla u|^2}{|u|^2}.$$

Theorem 2. *Let $l_k = \inf_{V\in\Phi_k(W_0^{1,2}(\Omega))} \sup_{u\in V} \mathcal{F}(u)$, then there exists $0 \neq u_k \in W_0^{1,2}(\Omega)$ solves*

$$\begin{cases} \Delta u_k = l_k u_k, \ \text{in } \Omega, \\ u_k = 0, \ \text{on } \partial\Omega \end{cases} \tag{6}$$

weakly. That is, for any $w \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla u_k \cdot \nabla w = l_k \int_\Omega u_k w.$$

Moreover, u_k is orthogonal to $\{u_j\}_{j=1}^{k-1}$.

Proof. For simplicity, we prove the case of $k = 1$ only.

Let $l_1 = \inf_{0\neq u\in W_0^{1,2}(\Omega)} \mathcal{F}(u)$. Let $w_n \in W_0^{1,2}(\Omega)$ such that $\|w_n\|_{L^2(\Omega)} = 1$ $\mathcal{F}(w_n) \to \lambda$. Then $\|w_n\|_{W^{1,2}(\Omega)} \leq C$ for some $C > 0$. Hence, since $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, we may assume that

$w_n \to u_1$ for some $u_1 \in L^2(\Omega)$. Moreover, since $\|\nabla w_n\|_{L^2(\Omega)} \le C$, we may assume that $\nabla w_n \to \psi$ in weak $L^2(\Omega)$-topology.

Then for $\rho \in C_c^\infty(\Omega)$,

$$\int_\Omega \psi\rho = \lim_{n\to\infty}\int_\Omega \nabla w_n \rho = -\lim_{n\to\infty}\int_\Omega w_n \nabla\rho = -\int_\Omega u_1\nabla\rho$$

Hence, u_1 has weak derivative ψ. Hence $u_1 \in W_0^{1,2}(\Omega)$.

Next, we would like to show that u_1 satisfies the EPDEs (6) weakly.

Fix $0 \ne \rho \in W_0^{1,2}(\Omega)$, $u_t = u_1 + t\rho$, then we must have

$$\frac{d}{dt}\mathcal{F}(u_t)|_{t=0} = 0.$$

Which, by a straightforward computation, implies that

$$\int_\Omega \nabla u \nabla\rho = l_1\int_\Omega u\rho.$$

□

Similarly,

Theorem 3. *Let $l_{k,N} = \inf_{V\in\Phi_k(W^{1,2}(\Omega))}\sup_{u\in V}\mathcal{F}(u)$, then there exists $0 \ne u_{k,N} \in W^{1,2}(\Omega)$ solves*

$$\begin{cases}\Delta u_{k,N} = l_{k,N}u_{k,N}, \ \textit{in } \Omega,\\ \partial_\nu u_{k,N} = 0, \ \textit{on } \partial\Omega\end{cases} \tag{7}$$

weakly. That is, for any $w \in W^{1,2}(\Omega)$,

$$\int_\Omega \nabla u_k\cdot\nabla w = l_k\int_\Omega u_k w.$$

Moreover, $u_{k,N}$ is orthogonal to $\{u_{j,N}\}_{j=1}^{k-1}$.

Remark 3. *In fact, $\lambda_k = l_k$ and $\lambda_{k,N} = l_{k,N}$. Moreover, one can take $e_k = \frac{u_k}{\|u_k\|_{L^2(\Omega)}}$ and $e_{k,N} = \frac{u_{k,N}}{\|u_{k,N}\|_{L^2(\Omega)}}$.*

To be continued...

References

[1] Lawrence C. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I., 2010.