Notes on string and string field theory

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Abstract

1 Bosonic string

1.1 Classical Action for Point Particles

In classical physics, the evolution of a theory is described by its field equations. Suppose we have an action

$$S = \int dtL$$

where $L = \frac{1}{2}m\dot{X}(t)^2 - V(X(t))$. Then $\delta S = 0$ gives Newton's law $m\ddot{X}(t) = -\partial V(X(t))/\partial X(t)$.

1.2 Classical Action for Relativistic Point Particles

Action is given by

$$S_0 = -\alpha \int ds, \tag{1}$$

where ds is given by

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu$$

where the metric $g_{\mu\nu}(X)$, with $\mu, \nu = 0, 1, \ldots, D-1$, describes the geometry of the background spacetime in which the theory is defined.

In particular, if we choose the geometry of our background spacetime to be Minkowskian then our metric can be written as

$$g_{\mu\nu}(X) \mapsto \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that, in a Minkowskian spacetime, the action becomes

$$S_0 = -m \int \sqrt{-\left(dX^0\right)^2 + \left(dX^1\right)^2 + \left(dX^2\right)^2 + \left(dX^3\right)^2}.$$

Proposition 1. *The action (1) remains unchanged if we replace the parameter τ by another parameter $\tau' = f(\tau)$, where f is monotonic.*

Since the square root function is a non-linear function, we would like to construct another action which does not include a square root in its argument. So we introduce an auxiliary field $e(\tau)$ and consider the equivalent action given by

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \left(e(\tau)^{-1}\dot{X}^2 - m^2 e(\tau)\right)$$

where $\dot{X}^2 \equiv g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$.

Consider the variation of $\tilde{S}_0$ with respect to the field $e(\tau)$

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \frac{\delta e}{e^2}\left(-\dot{X}^2 - m^2 e^2\right),$$

we get the field equations for $e(\tau)$

$$e^2 = -\frac{\dot{X}^2}{m^2} \Longrightarrow \quad e = \sqrt{\frac{-\dot{X}^2}{m^2}}. \tag{2}$$

Plugging the field equation for the auxiliary field back into the action $\tilde{S}_0$, we get $\tilde{S}_0 = S_0$.

Moreover, $\tilde{S}_0$ is that it is invariant under a reparametrization (diffeomorphism) of τ.

This invariance can be used to set the auxiliary field equal to unity, thereby simplifying the action. However, one must retain the field equations (2) for $e(\tau)$, in order to not lose any information. Hence, we have that

$$\dot{X}^2 + m^2 = 0 \tag{3}$$

which is the position representation of the mass-shell relation in relativistic mechanics.

1.2.1 Canonical Momenta

The canonical momentum, conjugate to the field $X^\mu(\tau)$, is defined by

$$P^\mu(\tau) = \frac{\partial L}{\partial \dot{X}^\mu}.$$

Hence (3) is nothing more than the mass-shell equation for a particle of mass m,

$$P^\mu P_\mu + m^2 = 0$$

1.2.2 Varying $\tilde{S}_0$ in an Arbitrary Background

Fixing $e(\tau) \equiv 1$, if we assume that the metric is not flat, and thus depends on its spacetime position, then varying $\tilde{S}_0$ with respect to $X^\mu(\tau)$ results in

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \left(-2\ddot{X}^\nu g_{\mu\nu}(X) - 2\partial_k g_{\mu\nu}(X)\dot{X}^k\dot{X}^\nu + \partial_\mu g_{k\nu}(X)\dot{X}^k\dot{X}^\nu\right)\delta X^\mu,$$

namely

$$\ddot{X}^\mu + \Gamma^\mu_{kl}\dot{X}^k\dot{X}^l = 0,$$

where Γ^μ_{kl} are the Christoffel symbols. These are the geodesic equations describing the motion of a free particle moving through a spacetime with an arbitrary background geometry.

1.3 Generalization to p-Branes

We now want to generalize the notion of an action for a point particle (0-brane), to an action for a p-brane. The generalization of $S_0 = -m\int ds$ to a p-brane in a $D(\geq p)$ dimensional background spacetime is given by

$$S_p = -T_p \int d\mu_p,$$

where T_p is the p-brane tension, which has units of mass/vol, and $d\mu_p$ is the $(p+1)$ dimensional volume element given by

$$d\mu_p = \sqrt{-\det\left(G_{\alpha\beta}(X)\right)}d^{p+1}\sigma$$

Where $G_{\alpha\beta}$ is the induced metric on the worldsurface, or worldsheet for $p = 1$, given by

$$G_{\alpha\beta}(X) = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu}(X) \quad \alpha,\beta = 0,1,\ldots,p$$

with $\sigma^0 \equiv \tau$ while $\sigma^1, \sigma^2, \ldots, \sigma^p$ are the p spacelike coordinates for the $p+1$ dimensional worldsurface mapped out by the p-brane in the background spacetime.

1.3.1 String action

If we assume that our background spacetime is Minkowski, then we have that

$$\begin{aligned} G_{00} &= \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu} \equiv \dot{X}^2 \\ G_{11} &= \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu} \equiv X'^2 \\ G_{10} = G_{01} &= \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu}. \end{aligned}$$

So our previous action reduces to (Nambu-Goto action)

$$S_{NG} = -T\int d\tau d\sigma \sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}^2\right)(X'^2)}.$$

Now, in order to get rid of the square root, we can introduce an auxiliary field $h_{\alpha\beta}(\tau,\sigma)$ (this really is another metric living on the worldsheet, which is different from the induced metric $G_{\alpha\beta}$)†, just like before with the auxiliary field $e(\tau)$. The resulting action is called the string sigma-model, or Polyakov action, and it is given by

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}\frac{\partial X^\mu}{\partial \alpha}\frac{\partial X^\nu}{\partial \beta}g_{\mu\nu}$$

where $h \equiv \det(h_{\alpha\beta})$.

Proposition 2. *The Polyakov action S_σ is equivalent to the Nambu-Goto action S_{NG}.*

1.4 Symmetries and Field Equations of the Bosonic String

1.4.1 Global symmetry

Poincare Transformation:
$\delta X^\mu(\tau,\sigma) = a^\mu_\nu X^\nu(\tau,\sigma) + b^\mu \;\; \delta h_{\alpha\beta}(\tau,\sigma) = 0$

1.4.2 Local Symmetries of the Bosonic String Theory Worldsheet

- Reparametrization invariance (also known as diffeomorphisms): This is a local symmetry for the worldsheet. The Polyakov action is invariant under the changing of the parameter σ to $\sigma' = f(\sigma)$:

$$X^\mu(\tau,\sigma) = X'\mu\left(\tau,\sigma'\right) \qquad \text{and} \qquad h_{\alpha\beta}(\tau,\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta}h'_{\gamma\delta}\left(\tau,\sigma'\right).$$

- Weyl Symmetry: Weyl transformations are transformations that change the scale of the metric,

$$h_{\alpha\beta}(\tau,\sigma) \mapsto h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma)$$

  while under a Weyl transformation, $\delta X^\mu(\tau,\sigma) = 0$.

  Recall that the stress-energy tensor is given by

$$T_{\alpha\beta} \equiv -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S_\sigma}{\delta h_{\alpha\beta}}.$$

  Thus, if we now restrict to a Weyl transformation we see that variation of the action becomes

$$\delta S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}(-2\phi)h^{\alpha\beta}T_{\alpha\beta}.$$

  transformation. Hence for a Weyl invariant classical theory,

$$h^{\alpha\beta}T_{\alpha\beta} = 0$$

  i.e. , the corresponding stress-energy tensor must be traceless.

Given a metric

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix}.$$

A diffeomorphism (or reparametrization) allows us assume that $h_{\alpha\beta}(X)$ is of the form $h(X)\eta_{\alpha\beta}$. Now, we can use a Weyl transformation to remove this function, i.e. we then have that $h_{\alpha\beta}(X) = \eta_{\alpha\beta}$. Consequently, if our theory is invariant under diffeomorphisms and Weyl transformations (there combinations are called conformal transformations), then the two-dimensional intrinsic metric, $h_{\alpha\beta}(X)$, can be "gauged" into the two-dimensional flat metric.

In terms of the gauge fixed flat metric, the Polyakov action becomes

$$S_\sigma = \frac{T}{2}\int d\tau d\sigma \left((\dot{X})^2 - (X')^2\right)$$

where $\dot{X} \equiv dX^\mu/d\tau$ and $X' \equiv dX^\mu/d\sigma$.

1.4.3 Field Equations for the Polyakov Action Let

The field equations for the fields $X^\mu(\tau,\sigma)$ on the worldsheet come from setting the variation of S_σ with respect to $X^\mu \mapsto X^\mu + \delta X^\mu$ equal to zero. This leads to

$$T\int d\tau d\sigma \left[\left(-\partial_\tau^2 + \partial_\sigma^2\right) X^\mu\right]\delta X^\mu + T\int d\sigma \dot{X}^\mu \delta X^\mu\Big|_{\partial\tau} - \left[T\int d\tau X'\delta X^\mu\Big|_{\sigma=\pi} + T\int d\tau X'\delta X^\mu\Big|_{\sigma=0}\right]$$

We set the variation of X^μ at the boundary of τ to be zero, i.e. $\delta X^\mu|_{\partial\tau} = 0$, and are left with the field equations for $X^\mu(\tau,\sigma)$ for the Polyakov action,

$$\left(-\partial_\tau^2 + \partial_\sigma^2\right) X^\mu - T\int d\tau \left[X'\delta X^\mu|_{\sigma=\pi} + X'\delta X^\mu|_{\sigma=0}\right].$$

- Closed string.
  We get the following field equations for the closed string

  $$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\tau,\sigma) = 0$$

  with the boundary conditions

  $$X^\mu(\tau,\sigma+n) = X^\mu(\tau,\sigma).$$

- Open Strings (Neumann Boundary Conditions).

  $$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\tau,\sigma) = 0$$

  with the boundary conditions

  $$\partial_\sigma X^\mu(\tau,\sigma+0) = \partial_\sigma X^\mu(\tau,\sigma+n) = 0$$

  Note that the Neumann boundary conditions preserve Poincaré invariance since

  $$\begin{aligned} \partial_\sigma\left(X'\mu\right)|_{\sigma=0,n} &= \partial_\sigma\left(a^\mu_\nu X^\nu + b^\mu\right)|_{\sigma=0,n} \\ &= a^\mu_\nu \partial_\sigma X^\nu|_{\sigma=0,n} \\ &= 0 \end{aligned}$$

- Open Strings (Dirichlet Boundary Conditions).

  $$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\tau,\sigma) = 0$$

  with boundary conditions

  $$X^\mu(\tau,\sigma=0) = X^\mu_0$$

and

$$X^\mu(\tau, \sigma = n) = X^\mu_n$$

Whereas the Neumann boundary conditions preserve Poincaré invariance, the Dirichlet boundary conditions do not since

$$\begin{aligned} (X'\mu)|_{\sigma=0,n} &= (a^\mu_\nu X^\nu + b^\mu)|_{\sigma=0,n} \\ &= a^\mu_\nu X^\nu_{0,n} + b^\mu \\ &\neq X^\mu_{0,n} \end{aligned}$$

Thus, under a Poincaré transformation the ends of the string actually change.

Finally, note that if we have Neumann boundary conditions on $p+1$ of the background spacetime coordinates and Dirichlet boundary conditions on the remaining $D-p+1$ coordinates, then the place where the string ends is a Dp-brane.

one must impose the field equations which result from setting the variation of S_σ with respect to $h^{\alpha\beta}$ equal to zero. These field equations are given by (see (2.35))

$$0 = T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X$$

and gauge fixing $h^{\alpha\beta}$ to be flat † we get that the field equations transform into the following two conditions

$$0 = T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right)$$

and

$$0 = T_{01} = T_{10} = \dot{X} \cdot X'.$$

1.4.4 Solving the Field Equations

By introducing light-cone coordinates for the worldsheet,

$$\sigma^\pm = (\tau \pm \sigma),$$

The field equations $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$ become

$$\partial_+ \partial_- X^\mu = 0. \tag{4}$$

While the field equations for the intrinsic worldsheet metric, $h_{\alpha\beta}$ become

$$\begin{aligned} T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \\ T_{--} = \partial_- X^\mu \partial_- X_\mu = 0 \end{aligned}$$

Solving (4), one has $X^\mu = \underbrace{X^\mu_R(\tau - \sigma)}_{\text{right mover}} + \underbrace{X^\mu_L(\tau + \sigma)}_{\text{left mover}}$

- Closed String.
  Applying the closed string boundary conditions $X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma)$ gives the particular solution (mode expansion) for the left and right movers as

  $$\begin{aligned} X^\mu_R = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2(\tau - \sigma)p^\mu + \tfrac{i}{2}l_s \textstyle\sum_{n\neq 0} \tfrac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}, \\ X^\mu_L = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2(\tau + \sigma)p^\mu + \tfrac{i}{2}l_s \textstyle\sum_{n\neq 0} \tfrac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}, \end{aligned}$$

  where x^μ is a constant (called the center of mass of the string), p^μ is a constant (called the total momentum of the string), l_s is the string length (also a constant), $T = \frac{1}{2n\alpha'}$ and $\alpha' = \frac{1}{2}l_s^2$.
  Now, since X^μ must be real, i.e. $(X^\mu)^* = X^\mu$, we get that x^μ and p^μ are real along with

  $$\begin{aligned} \alpha^\mu_{-n} = (\alpha^\mu_n)^* \\ \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*. \end{aligned}$$

Furthermore, from the definition of the canonical momentum, $\overset{\dagger}{P^{\mu}}(\tau,\sigma)$, we can see that the mode expansion of the canonical momentum on the worldsheet is given by

$$\begin{aligned} P^{\mu}(\tau,\sigma) &= \frac{\delta L}{\delta \dot{X}^{\mu}} = T\dot{X}^{\mu} = \frac{\dot{X}^{\mu}}{\pi l_s^2} \\ &= \frac{p^{\mu}}{\pi} + \frac{1}{\pi l_s}\sum_{n\neq 0}\left(\alpha_n^{\mu} e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^{\mu} e^{-2in(\tau+\sigma)}\right). \end{aligned}$$

Now, it can be shown that the field and its canonical momentum satisfy the following Poisson bracket relations

$$\begin{aligned} &\{P^{\mu}(\tau,\sigma), P^{\nu}(\tau,\sigma')\}_{P.B.} = 0, \\ &\{X^{\mu}(\tau,\sigma), X^{\nu}(\tau,\sigma')\}_{P.B} = 0, \\ &\{P^{\mu}(\tau,\sigma), X^{\nu}(\tau,\sigma')\}_{P.B.} = \eta^{\mu\nu}\delta\left(\sigma-\sigma'\right), \end{aligned}$$

which implies

$$\begin{aligned} &\{\alpha_m^{\mu}, \alpha_n^{\nu}\}_{P.B.} = \{\tilde{\alpha}_m^{\mu}, \tilde{\alpha}_n^{\nu}\} = im\eta^{\mu\nu}\delta_{m,-n} \\ &\{\alpha_m^{\mu}, \alpha_n^{\nu}\}_{P.B.} = 0 \\ &\{p^{\mu}, x^{\nu}\}_{P.B} = \eta^{\mu\nu} \end{aligned}$$

- Open String(Neumann Boundary Conditions).

- Open String(Dirichlet Boundary Conditions).

1.4.5 Symmetry and Charge

Associated to any global symmetry of a system, in our case the worldsheet, there exists a conserved current, j^{μ}, and a conserved charge, Q, i.e.

$$\begin{aligned} &\partial_{\alpha} j^{\alpha} = 0 \\ &\tfrac{d}{d\tau}Q = \tfrac{d}{d\tau}\left(\int d\sigma j^{0}\right) = 0 \end{aligned}$$

where the integral in the expression for the charge is taken over the spacelike coordinates, which in our case is just σ.

- For Translation.

$$\delta X^{\mu} == b^{\mu}\left(\sigma^{\alpha}\right),$$

$$j_{\mu}^{\alpha} = -T\partial^{\alpha}X_{\mu}.$$

  The corresponding charge Q is given by

$$\begin{aligned} p^{\nu} &= \int d\sigma j^{0\nu} \\ &= \int_0^{\pi} d\sigma T\dot{X}^{\nu} \\ &= \int_0^{\pi} d\sigma P^{\mu} \end{aligned}$$

- Lorentz Transformations.

$$\delta X^{\mu} = a_k^{\mu} X^{k}.$$

$$j_{\alpha}^{\mu\nu} = -\frac{T}{2}\left(X^{\mu}\partial_{\alpha}X^{\nu} - X^{\nu}\partial_{\alpha}X^{\mu}\right)$$

1.4.6 The Hamiltonian and Energy-Momentum Tensor

Worldsheet time evolution is generated by the Hamiltonian which is defined by

$$H = \int_{\sigma=0}^{\pi} d\sigma \left(\dot{X}_\mu P^\mu - \mathcal{L}\right).$$

In our case

$$H = \frac{T}{2}\int_{\sigma=0}^{\pi} d\sigma \left(\dot{X}^2 + X'^2\right).$$

For a closed string theory the Hamiltonian becomes

$$H = \sum_{n=-\infty}^{\infty} \left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right).$$

Since

$$\begin{aligned} T_{--} &= \left(\partial_- X_R^\mu\right)^2, \\ T_{++} &= \left(\partial_+ X_L^\mu\right)^2, \\ T_{-+} &= T_{+-} = 0, \end{aligned}$$

One has

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}.$$

Here

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$

. Similarly,

$$T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau-\sigma)}.$$

Here

$$\tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n.$$

For a closed string we have that

$$H = 2\left(L_0 + \tilde{L}_0\right).$$

1.4.7 Classical Mass Formula for a Bosonic String

Recall the mass-energy relation,

$$M^2 = -p^\mu p_\mu.$$

Now, for our Bosonic string theory we have that

$$p^\mu = \int_{\sigma=0}^{\pi} d\sigma P^\mu = T\int_{\sigma=0}^{\pi} d\sigma \dot{X}^\mu = \begin{cases} \frac{2\alpha_0^\mu}{l_s} & \text{for a closed string,} \\ \frac{\alpha_0^\mu}{l_s} & \text{for an open string.} \end{cases}$$

Hence

$$p^\mu p_\mu = \begin{cases} \frac{2\alpha_0^2}{\alpha'} & \text{for a closed string,} \\ \frac{\alpha_0^2}{2\alpha'} & \text{for an open string.} \end{cases}$$

Here $\alpha' = l_s^2/2$.

Hence for open string,

$$\begin{aligned} 0 = L_0 &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n \\ &= \left(\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\right) + \underbrace{\alpha' p^\mu p_\mu}_{=M^2}. \end{aligned}$$

For closed string,

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right).$$

Remark 1. *These are the mass-shell conditions for open and closed strings and they tell you the mass corresponding to a certain classical string state. They are only valid classically since the expressions for $T_{\alpha\beta}$ and H, in which they were derived, are only valid classically. In the quantized theory they will get altered a bit.*

1.4.8 Witt Algebra (Classical Virasoro Algebra)

The set of elements $\{L_m\}$ forms an algebra whose multiplication is given by

$$\{L_m, L_n\}_{P.B.} = i(m-n)L_{m+n}$$

where $\{\cdot,\cdot\}_{P.B.}$ is the Poisson bracket.This algebra is called the Witt algebra or the classical Virasoro algebra. A good question to ask is what is the physical meaning of the L_m's?

Previously, we fixed the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$. However, this does not completely gauge fix the diffeomorphism and Weyl symmetries. Consider the transformations given by

$$\begin{aligned}\delta_D\eta^{\alpha\beta} &= -\left(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha\right),\\ \delta_W\eta^{\alpha\beta} &= \Lambda\eta^{\alpha\beta},\end{aligned}$$

where ξ^α is an infinitesimal parameter of reparametrization, Λ is an infinitesimal parameter for Weyl rescaling, $\delta_D\eta^{\alpha\beta}$ gives the variation of the metric under reparametrization and $\delta_W\eta^{\alpha\beta}$ give the variation under a Weyl rescaling. If we combine these two transformations we get

$$\left(\delta_D + \delta_W\right)\eta^{\alpha\beta} = \left(-\partial^\alpha\xi^\beta - \partial^\beta\xi^\alpha + \Lambda\eta^{\alpha\beta}\right).$$

Now, what is the most general solution for ξ and Λ such that the above equation is zero? If we can find these then it means that we have found additional symmetries for our system, which correspond to reparametrizations which are also Weyl rescalings, i.e. conformal transformations. In terms of the light-cone coordinates, the equation to be solved becomes

$$\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha = \Lambda\eta^{\alpha\beta}.$$

- $\alpha = \beta = +$: Noting that $\eta^{++} = 0$ we have to solve

$$\begin{aligned}\partial^+\xi^+ + \partial^+\xi^+ &= \Lambda\eta^{++}\\ \Longrightarrow \partial^+\xi^+ &= 0.\end{aligned}$$

- $\alpha = \beta = -$: we get $\Longrightarrow \partial^-\xi^- = 0$.
- $\alpha = +, \beta = -$: we have

$$\partial^+\xi^- + \partial^-\xi^+ = -2\Lambda.$$

So, local transformations which satisfy

$$\begin{aligned}\delta\sigma^+ &= \xi^+\left(\sigma^+\right)\\ \delta\sigma^- &= \xi^-\left(\sigma^-\right)\\ \Lambda &= \partial^-\xi^+ + \partial^+\xi^-\end{aligned}$$

leave our theory invariant.

Note that the infinitesimal generators for the transformations $\delta\sigma^\pm = \xi^\pm$are given by

$$V^\pm = \frac{1}{2}\xi^\pm\left(\sigma^\pm\right)\frac{\partial}{\partial\sigma^\pm}$$

and a complete basis for these transformations is given by

$$\xi_n^\pm(\sigma^\pm) = e^{2in\sigma^\pm}, n \in \mathbb{Z}.$$

The corresponding generators $V_n^\pm$ give two copies of the Witt algebra L_m, while in the case of the open string there is just one copy of the Witt algebra, and the infinitesimal generators are given by

$$V_n^\pm = e^{in\sigma^+}\frac{\partial}{\partial\sigma^+} + e^{in\sigma^-}\frac{\partial}{\partial\sigma^-}, \quad \text{n} \in \mathbb{Z}.$$

1.5 Canonical Quantization of the Bosonic String

In the canonical quantization procedure, we quantize the theory by changing Poisson brackets to commutators,

$$\{\cdot,\cdot\}_{P.B.} \mapsto i[\cdot,\cdot],$$

and we promote the field X^μ to an operator in our corresponding Hilbert space. This is equivalent to promoting the modes α, the constant x^μ and the total momentum p^μ to operators. In particular, for the modes α^μ_m, we have that (here and usually in the sequel we are dropping the i factor)

$$\begin{aligned}
[\hat{\alpha}^\mu_m, \hat{\alpha}^\nu_n] &= m\eta^{\mu\nu}\delta_{m,-n}, \\
\left[\hat{\tilde{\alpha}}^\mu_m, \hat{\tilde{\alpha}}^\nu_n\right] &= m\eta^{\mu\nu}\delta_{m,-n}, \\
\left[\hat{\alpha}^\mu_m, \hat{\tilde{\alpha}}^\nu_n\right] &= 0,
\end{aligned}$$

where the $\hat{\alpha}$'s on the RHS are realized as operators in a Hilbert space. If we define new operators as $\hat{a}^\mu_m \equiv \frac{1}{\sqrt{m}}\hat{\alpha}^\mu_m$ and $\hat{a}^{\mu\dagger}_m \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_{-m}$, for $m > 0$, then they clearly satisfy

$$\left[\hat{a}^\mu_m, \hat{a}^{\nu\dagger}_n\right] = \left[\hat{\tilde{a}}^\mu_m, \hat{\tilde{a}}^{\nu\dagger}_n\right] = \eta^{\mu\nu}\delta_{m,n} \quad \text{for } m,n > 0$$

This looks like the same algebraic structure as the algebra constructed from the creation/annihilation operators of quantum mechanics, except that for $\mu = \nu = 0$ we get a negative sign, due to the signature of the metric,

$$\left[\hat{a}^0_m, \hat{a}^{0\dagger}_n\right] = \eta^{00}\delta_{m,n} = -\delta_{m,n}.$$

We will see later that this negative sign in the commutators leads to the prediction of negative norm physical states, or ghost states. which is incorrect.

Next, we define the ground state $|0\rangle$, as the state which is annihilated by all of the lowering operators $\hat{a}^\mu_m$,

$$\hat{a}^\mu_m|0\rangle = 0 \quad \text{for } m > 0$$

Also, physical states are states that are constructed by acting on the ground state with the raising operators $\hat{a}^{\mu\dagger}_m$

$$|\phi\rangle = \hat{a}^{\mu_1\dagger}_{m_1}\hat{a}^{\mu_2\dagger}_{m_2}\cdots\hat{a}^{\mu_n\dagger}_{m_n}|0;k^\mu\rangle,$$

which are also eigenstates of the momentum operator $\hat{p}^\mu$,

$$\hat{p}^\mu|\phi\rangle = k^\mu|\phi\rangle.$$

To prove the claim of negative norm states, consider the state $|\psi\rangle = \hat{a}^{0\dagger}_m|0;k^\mu\rangle$, for $m > 0$, then we have that

$$\begin{aligned}
\||\psi\rangle\|^2 &= \left\langle 0\left|\hat{a}^0_m\hat{a}^{0\dagger}_m\right|0\right\rangle \\
&= \left\langle 0\left|\left[\hat{a}^0_m, \hat{a}^{0\dagger}_m\right]\right|0\right\rangle \\
&= -\langle 0 \mid 0\rangle.
\end{aligned}$$

1.5.1 Virasoro Algebra

We have seen that when we quantize our bosonic string theory the modes α become operators. This then implies that the generators L_m will also become operators. However, one must be careful because we simply cannot just say that $\hat{L}_m$ is given by

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n, \quad \text{(wrong!)}$$

We must, as in QFT, normal order the operators and thus we define $\hat{L}_m$ to be

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n: .$$

Note that normal ordering ambiguity only arises for the case when $m = 0$, i.e for the operator $\hat{L}_0$. In normal ordering, we have that $\hat{L}_0$ is given by

$$\hat{L}_0 = \frac{1}{2}\hat{\alpha}_0^2 + \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n.$$

We introduce normal ordering due to the tact that there is an ordering ambiguity arising from the commutation relations of the operators $\hat{\alpha}$ and $\hat{\tilde{\alpha}}$. When we commute the operators past each other we pick up extra constants. So, how do we know what order to put the operators in? The answer is that we do not. We simply take the correct ordering to be normal ordering. Also, in terms of the commutation relations for the operators $\hat{\alpha}$, we get that the commutation relations for the operators $\hat{L}_m$ are given by

$$\left[\hat{L}_m, \hat{L}_n\right] = (m-n)\hat{L}_{m+n} + \frac{c}{12} m\left(m^2 - 1\right)\delta_{m,-n}$$

where c is called the central charge.

We will see that, in the bosonic string theory, c is equal to the dimension of the spacetime where the theory lives, and in order to no longer have non-negative norm states it must be that $c = 26$. Also, note that for $m = -1, 0, 1$ the c term drops out and we get a subalgebra of $SL(2, \mathbb{R})$, i.e the set $\left\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\right\}$ along with the relations

$$\left[\hat{L}_m, \hat{L}_n\right] = (m-n)\hat{L}_{m+n}$$

becomes an algebra which is isomorphic to $SL(2, \mathbb{R})$.

1.5.2 Physical states

Classically we have seen that $L_0 = 0$ since to the vanishing of the stress-energy tensor implies that $L_m = 0$ for all m, but when we quantize the theory we cannot say that $\hat{L}_0 = 0$, or equivalently $\hat{L}_0|\phi\rangle = 0$ for all physical states, follows from this as well because when we quantize the theory we have to normal order the operator $\hat{L}_0$ and so we could have some arbitrary constant due to this normal ordering. Thus, after quantizing we can at best say that for an open string the vanishing of the L_0 constraint transforms to

$$\left(\hat{L}_0 - a\right)|\phi\rangle = 0$$

where a is a constant. This is called the mass-shell condition for the open string. While for a closed string we have that

$$\begin{aligned} \left(\hat{L}_0 - a\right)|\psi\rangle = 0 \\ \left(\hat{\tilde{L}}_0 - a\right)|\psi\rangle = 0 \end{aligned} \tag{5}$$

where $\hat{\tilde{L}}$ is the operator corresponding to the classical generator $\tilde{L}$.

Normal ordering also adds correction terms to the mass formula. For an open string theory, the mass formula becomes

$$\alpha' M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : -a = \hat{N} - a,$$

where we have defined the number operator $\hat{N}$ as

$$\hat{N} = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n := \sum_{n=1}^{\infty} n : \hat{a}_n^{\dagger} \cdot \hat{a}_n : .$$

We can use the number operator to compute the mass spectrum,

$$\begin{aligned} &\alpha' M^2 = -a \quad (\text{ ground state n} = 0), \\ &\alpha' M^2 = -a + 1 \quad \left(\text{first excited state n} = 1\right), \\ &\alpha' M^2 = -a + 2 \quad \left(\text{second excited state n} = 2\right). \end{aligned}$$

For a closed string we have the mass formula

$$\frac{4}{\alpha'}M^2 = \sum_{n=1}^{\infty} :\hat{\alpha}_{-n}\cdot\hat{\alpha}_n: -a = \sum_{n=1}^{\infty} :\hat{\tilde{\alpha}}_{-n}\cdot\hat{\tilde{\alpha}}_n: -a,$$

or,

$$\frac{4}{\alpha'}M^2 = \hat{N} - a = \hat{\tilde{N}} - a.$$

Also, note that if we subtract the left moving physical state condition from the right moving physical state condition (**??**), we get that

$$\left(\hat{L}_0 - a - \hat{\tilde{L}} + a\right)|\phi\rangle = 0,$$

which implies that

$$\left(\hat{L}_0 - \hat{\tilde{L}}_0\right)|\phi\rangle = 0,$$

which in turn implies that

$$\hat{N} = \hat{\tilde{N}}.$$

Classically we have that $L_m = 0$ for all m, which we know does not hold for $\hat{L}_0$, but what about the operators $\hat{L}_m$ for $m \neq 0$? Well, if $\hat{L}_m|\phi\rangle = 0$ for all $m \neq 0$ then we would have that (if we take n in such a way that $n + m \neq 0$),

$$\left[\hat{L}_m, \hat{L}_n\right]|\phi\rangle = 0$$

But when we plug in the commutation relations we get

$$(m-n)\hat{L}_{n+m}|\phi\rangle + \frac{c}{12}m\left(m^2-1\right)\delta_{m,-n}|\phi\rangle = 0$$

and since the first term vanishes (because we are assuming $\hat{L}_m|\phi\rangle = 0$ for all $m \neq 0$) we see that if $c \neq 0$ then it must be that either $m = -1, m = 0$ or $m = 1$. Thus, if we want to have $\hat{L}_m|\phi\rangle = 0$ for all m then we must restrict our Virasoro algebra to only $\left\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\right\}$. Instead of doing this we will only impose that $\hat{L}_m|\phi\rangle = 0 = \langle\phi|\hat{L}_m^\dagger$ for $m > 0$. Physical states are then characterized by

$$\hat{L}_{m>0}|\phi\rangle = 0 = \langle\phi|\hat{L}_{m>0}^\dagger$$

and the mass-shell condition

$$\left(\hat{L}_0 - a\right)|\phi\rangle = 0.$$

1.6 Removing Ghost States and Light-Cone Quantization

Previously, we have the physical states $|\phi\rangle$ were defined as states, $|\phi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\hat{a}_{m_2}^{\mu_2\dagger}\cdots\ \hat{a}_{m_n}^{\mu_n\dagger}|0;k^\mu\rangle$, which obeyed the following two constraints

$$\left(\hat{L}_0 - a\right)|\phi\rangle = 0,$$
$$\hat{L}_{m>0}|\phi\rangle = 0.$$

We also saw that there were certain physical states whose norm was less than zero, a trait that no physical state can have. However, as was already mentioned, we can get rid of these negative norm physical states by constraining the constant a, and by also constraining the central charge of the Virasoro algebra.

1.6.1 Spurious States

A state, $|\psi\rangle$, is said to be spurious if it satisfies the mass-shell condition,

$$\left(\hat{L}_0 - a\right)|\psi\rangle = 0$$

and

$$\langle\phi \mid \psi\rangle = 0, \quad \forall \text{ physical states } |\phi\rangle.$$

In general, it follows from the definition of a spurious state that a spurious state can be written as (Also note that $\hat{L}_{-n}^{\dagger} = \hat{L}_n$)

$$|\psi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n} |\chi_n\rangle ,$$

where $|\chi_n\rangle$ is some state which satisfies the, now modified, mass-shell condition given by

$$\left(\hat{L}_0 - a + n\right) |\chi_n\rangle = 0.$$

Now, since any $\hat{L}_{-n}$, for $n \geq 1$, can be written as a combination of $\hat{L}_{-1}$ and $\hat{L}_{-2}$ the general expression for a spurious state (5.4) can be simplified to

$$|\psi\rangle = \hat{L}_{-1} |\chi_1\rangle + \hat{L}_{-2} |\chi_2\rangle$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ are called level 1 and level 2 states, respectively, and they satisfy the mass-shell conditions $\left(\hat{L}_0 - a + 1\right) |\chi_1\rangle = 0$ and $\left(\hat{L}_0 - a + 2\right) |\chi_2\rangle = 0$, respectively.

For example, consider the level 3 state given by $|\psi\rangle = \hat{L}_{-3} |\chi_3\rangle$. We have that $\Big(\hat{L}_0 - \ a + 3) |\chi_3\rangle = 0$ as well as $\hat{L}_{-3} = \left[\hat{L}_{-1}, \hat{L}_{-2}\right]$ which gives us that

$$\begin{aligned}
\hat{L}_{-3} |\chi_3\rangle &= \left[\hat{L}_{-1}, \hat{L}_{-2}\right] |\chi_3\rangle \\
&= \hat{L}_{-1}\hat{L}_{-2} |\chi_3\rangle - \hat{L}_{-2}\hat{L}_{-1} |\chi_3\rangle \\
&= \hat{L}_{-1} \left(\hat{L}_{-2} |\chi_3\rangle\right) + \hat{L}_{-2} \left(\hat{L}_{-1} |-\chi_3\rangle\right) .
\end{aligned}$$

Since a spurious state $|\psi\rangle$ is perpendicular to all physical states, if we require that $|\psi\rangle$, then

$$\||\psi\rangle\|^2 = \langle\psi \mid \psi\rangle = 0.$$

Thus, we have succeeded in constructing physical states whose norm is zero and these are precisely the states we need to study in order to get rid of the negative norm physical states in our bosonic string theory.

1.6.2 Removing the Negative Norm Physical States

We want to study physical spurious states in order to determine the values of a and c that project out the negative norm physical states, also called ghost states. So, in order to find the corresponding a value we should start with a level 1 physical spurious state,

$$|\psi\rangle = \hat{L}_{-1} |\chi_1\rangle$$

with $|\chi_1\rangle$ satisfying $\left(\hat{L}_0 - a + 1\right) |\chi_1\rangle = 0$ and $\hat{L}_{m>0} |\chi_1\rangle = 0$, where the last relation comes because we have assumed $|\psi\rangle$ to be physical.

Now, if $|\psi\rangle$ is physical, which we have assumed, then it must satisfy the mass-shell condition for physical states,

$$\left(\hat{L}_0 - a\right) |\psi\rangle = 0,$$

along with the condition

$$\hat{L}_{m>0}|\psi\rangle = 0.$$

So, if $\hat{L}_{m>0}|\psi\rangle = 0$ then this holds for, in particular, the operator $\hat{L}_1$, i.e. $\hat{L}_1|\psi\rangle = 0$ which implies that $a = 1$.

Next, in order to determine the appropriate value of c for spurious physical states we need to look at a level 2 spurious state. Note that a general level 2 spurious state is given by

$$|\psi\rangle = \left(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}\right) |\chi_2\rangle ,$$

where γ is a constant, that will be fixed to insure that $|\psi\rangle$ has a zero norm (i.e. physical), and $|\chi_2\rangle$ obeys the relations,

$$\left(\hat{L}_0 - a + 2\right)|\chi_2\rangle = 0,$$

and

$$\hat{L}_{m>0}\,|\chi_2\rangle = 0.$$

Similarly, $\hat{L}_1|\psi\rangle = 0$ implies that $\gamma = \frac{3}{2}$, and $\hat{L}_2|\psi\rangle = 0$ implies that $c = 26$. So, if we want to project out the negative norm physical states (ghost states) then we must restrict the values of a, γ and c to $1, 3/2$ and 26 , respectively. Also, note that since the central charge c is equivalent to the dimension of the background spacetime for our bosonic string theory, then our theory is only physically acceptable for the case that it lives in a space of 26 dimensions. The $a = 1, c = 26$ bosonic string theory is called critical, and the critical dimension is 26 . Finally, there can exist bosonic string theories with non-negative norm physical states for $a \leq 1$ and $c \leq 25$, which are called non-critical.

1.6.3 Light-Cone Gauge Quantization of the Bosonic String

Now, we will quantize the theory in a different manner that will no longer have negative norm physical states at the cost of not being manifestly Lorentz invariant. We can fix this however, at the cost of, once again, constraining the constants a and c. We will no longer use the hat overtop of operators, i.e. we will write A for $\hat{A}$, unless there is chance for confusion.

The light-cone coordinates for the background spacetime, $X^+$and X^-, are defined as

$$\begin{aligned} X^+ &\equiv \tfrac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) \\ X^- &\equiv \tfrac{1}{\sqrt{2}}\left(X^0 - X^{D-1}\right) \end{aligned}$$

So, the spacetime coordinates become the set $\left\{X^-, X^+, X^i\right\}_{i=1}^{D-2}$.

Note that since we are treating two coordinates of spacetime differently from the rest, namely X^0 and X^{D-1}, we have lost manifest Lorentz invariance and so our Lorentz symmetry $SO(1, D-1)$ becomes $SO(D-2)$.

1.7 CFT

1.8 When target space is a Riemannian manifold

1.8.1 Target space is $\mathbb{R}$

We formulate the theory on the cylinder $\Sigma = \mathbb{R} \times S^1$ where $\mathbb{R}$ is parametrized by the time t and S^1 is parametrized by the spatial coordinate s of period $2\pi, s \equiv s + 2\pi$. The action for the scalar field $x = x(t, s)$ is given by

$$S = \frac{1}{2\pi}\int_\Sigma L dt ds = \frac{1}{4\pi}\int_\Sigma \left((\partial_t x)^2 - (\partial_s x)^2\right) dt ds.$$

The action is invariant under the shift in x

$$\delta x = \alpha$$

where α is a constant. It's conserved charge is given by

$$p = \frac{1}{2\pi}\int_{S^1} j^t ds$$

with $j^t = \partial_t x$.

The action is also invariant under worldsheet space-time translations

$$\delta_\alpha x = \alpha^\mu \partial_\mu x$$

The conserved currents are

$$\left\{ \begin{array}{l} T^t_t = \frac{1}{2}\left((\partial_t x)^2 + (\partial_s x)^2\right), \\ T^s_t = -\partial_s x \partial_t x, \end{array} \right. \qquad \left\{ \begin{array}{l} T^t_s = \partial_s x \partial_t x \\ T^s_s = -\frac{1}{2}\left((\partial_t x)^2 + (\partial_s x)^2\right) \end{array} \right.$$

and the conserved charges are

$$H=\frac{1}{2\pi}\int_{S^1}T_t^t ds=\frac{1}{2\pi}\int_{S^1}\frac{1}{2}\left((\partial_t x)^2+(\partial_s x)^2\right)ds$$
$$P=\frac{1}{2\pi}\int_{S^1}T_s^t ds=\frac{1}{2\pi}\int_{S^1}\partial_t x\partial_s x ds.$$

Quantization. One has mode expansion

$$\hat{H}=\hat{H}_0+\sum_{n=1}^{\infty}\hat{H}_n,$$

where $\hat{H}_0=\frac{1}{2}\hat{p}_0^2$, $\hat{H}_n=\hat{\alpha}_{-n}\hat{\alpha}_n+\hat{\tilde{\alpha}}_{-n}\hat{\tilde{\alpha}}_n+n$ (n comes from communicator).
The target space momentum is simply

$$p=\frac{1}{2\pi}\int_{S^1}\dot{x}ds=\dot{x}_0=p_0,$$

and there is a momentum eigenstate $|k\rangle_0$ for each k

$$p_0|k\rangle_0=k|k\rangle_0$$

This is also the energy $k^2/2$ eigenstate of the Hamiltonian

$$H_0=\frac{1}{2}p_0^2,$$

We define $|0\rangle_n$ as the vector annihilated by α_n and $\widetilde{\alpha}_n$. This is a ground state for the Hamiltonian H_n, with energy n. A general energy eigenstate is constructed by multiplying powers of creation operators α_{-n} and $\widetilde{\alpha}_{-n}$ acting on $|0\rangle$.

The Hilbert space of the total system is a tensor product of the Hilbert spaces of these constituent theories. Let us define the state

$$|0;k\rangle:=|k\rangle_0\otimes\bigotimes_{n=1}^{\infty}|0\rangle_n.$$

The state

$$\prod_{n=1}^{\infty}(\alpha_{-n})^{m_n}(\widetilde{\alpha}_{-n})^{\widetilde{m}_n}|k\rangle$$

has the following worldsheet energy and momentum

$$H=\frac{k^2}{2}+\sum_{n=1}^{\infty}n\left(m_n+\widetilde{m}_n\right)-\frac{1}{12}$$

$$P=\sum_{n=1}^{\infty}n\left(-m_n+\tilde{m}_n\right)$$

and also has the target space momentum $p=k$.

The state $|0;0\rangle$ is the unique ground state with the ground state energy

$$E_0=-\frac{1}{12}$$

and target space momentum $p=0$.

Vertex Operator One computes

$$x(t_1,s_1)x(t_2,s_2)=:x(t_1,s_1)x(t_2,s_2):-it_1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left((z_2/z_1)^n+(\widetilde{z}_2/\widetilde{z}_1)^n\right)$$

where $z_j = \mathrm{e}^{i(t_j - s_j)}$ and $\tilde{z}_j = \mathrm{e}^{i(t_j + s_j)}$.

From now on, we assume an infinitesimal Wick rotation $t \to \mathrm{e}^{-i\epsilon} t$ with $\epsilon > 0$ (the complete Wick rotation $\epsilon = \pi/2$ would lead to $\widetilde{z}_i = \bar{z}_i$). If $t_1 > t_2$, we have $|z_2/z_1| < 1, |\widetilde{z}_2/\widetilde{z}_1| < 1$, and the sum is convergent to $-\frac{1}{2}\log(1 - z_2/z_1) - \frac{1}{2}\log(1 - \tilde{z}_2/\tilde{z}_1)$. This convergence shows that $\mathrm{T}\left[x(t_1, s_1)\, x(t_2, s_2)\right] =: x(t_1, s_1)\, x(t_2, s_2) : -\frac{1}{2}\log\left[(z_1 - z_2)(\widetilde{z}_1 - \widetilde{z}_2)\right]$ where $\mathrm{T}\left[A(t_1, s_1)\, B(t_2, s_2)\right]$ is the time ordered product, which is $A(1)B(2)$ if $t_1 > t_2$ and $B(2)A(1)$ if $t_2 > t_1$.

Partition Function. Consider $Z(\tau_1, \tau_2) = \mathrm{Tr}\, \mathrm{e}^{-2\pi i \tau_1 P} \mathrm{e}^{-2\pi \tau_2 H}$. Let us define

$$\begin{aligned} H_R &:= \tfrac{1}{2}(H - P) = \tfrac{1}{4}p_0^2 + \textstyle\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \tfrac{1}{24} \\ H_L &:= \tfrac{1}{2}(H + P) = \tfrac{1}{4}p_0^2 + \textstyle\sum_{n=1}^{\infty} \widetilde{\alpha}_{-n}\widetilde{\alpha}_n - \tfrac{1}{24}. \end{aligned}$$

Then the partition function can be written as

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \mathrm{Tr}\, \mathrm{e}^{2\pi i \tau H_R} \mathrm{e}^{-2\pi i \bar{\tau} H_L}, \\ &= \mathrm{Tr}\, q^{H_R} \bar{q}^{H_L}, \end{aligned}$$

where

$$\tau = \tau_1 + i\tau_2.$$

Then $Z(\tau, \bar{\tau}) = (q\bar{q})^{-1/24}\, \mathrm{Tr}_{\mathcal{H}_0} (q\bar{q})^{p_0^2/4} \prod_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n^R}\, q^{\alpha_{-n}\alpha_n}\, \mathrm{Tr}_{\mathcal{H}_n^L}\, \bar{q}^{\widetilde{\alpha}_{-n}\widetilde{\alpha}_n}$.

Moreover,

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_n^R}\, q^{\alpha_{-n}\alpha_n} &= \sum_{k=0}^{\infty} q^{nk} = \frac{1}{1 - q^n}, \\ \mathrm{Tr}_{\mathcal{H}_n^L}\, \bar{q}^{\widetilde{\alpha}_{-n}\widetilde{\alpha}_n} &= \frac{1}{1 - \bar{q}^n}, \\ \mathrm{Tr}_{\mathcal{H}_0} (q\bar{q})^{p_0^2/4} = \mathrm{Tr}_{\mathcal{H}_0}\, \mathrm{e}^{-2\pi\tau_2 H_0} &= V \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \mathrm{e}^{-2\pi\tau_2\left(\frac{1}{2}p^2\right)} = \frac{V}{2\pi}\frac{1}{\sqrt{\tau_2}}. \end{aligned}$$

In the last part, V stands for the cut-off volume in order to make the partition function finite. Putting all these factors together we obtain

$$\begin{aligned} Z(\tau, \bar{\tau}) &= (q\bar{q})^{-1/24} \frac{V}{2\pi}\frac{1}{\sqrt{\tau_2}} \prod_{n=1}^{\infty} \left| \frac{1}{1 - q^n} \right|^2 \\ &= \frac{V}{2\pi}\frac{1}{\sqrt{\tau_2}} |\eta(\tau)|^{-2}, \end{aligned}$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

One can see that the partition function is invariant under the differomorphisms on T^2 acting on τ as

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}$$

1.8.2 Target space is S_R^1

Unlike in case of the real line, since the circle has discrete Fourier modes (as we have studied in Sec. 10.1.1) the target space momentum is quantized in units of $1/R$:

$$p = l/R, l \in \mathbb{Z}.$$

Also, the target space coordinate x is not single-valued but is a periodic variable of period $2\pi R$. This means that there are topologically non-trivial field configurations in the theory which are classified by the winding number m defined by

$$x(s + 2\pi) = x(s) + 2\pi m R.$$

As we have seen, the conserved current for the momentum is

$$\begin{cases} j^t = \partial_t x \\ j^s = -\partial_s x \end{cases}$$

One can find another current

$$\begin{cases} j^t_w = \partial_s x \\ j^s_w = -\partial_t x \end{cases}$$

which satisfies the "conservation equation" $\partial_\mu j^\mu_w = 0$. The corresponding "charge" is

$$w = \frac{1}{2\pi}\int_{S^1} j^t_w ds = \frac{1}{2\pi}(x(2\pi) - x(0)) = mR$$

in the sector with winding number m. Thus, w is the topological charge that counts the winding number.

The Hilbert space $\mathcal{H}$ is decomposed into sectors labelled by two integers - momentum l and winding number m :

$$\mathcal{H} = \bigoplus_{(l,m)\in\mathbb{Z}\oplus\mathbb{Z}} \mathcal{H}_{(l,m)}$$

The subspace $\mathcal{H}_{(l,m)}$ is the space with $p = l/R$ and $w = mR$ and contains a basic element

$$|0; l, m\rangle$$

which is annihilated by α_n and $\widetilde{\alpha}_n$ with $n > 0$. The space $\mathcal{H}_{(l,m)}$ is constructed by acting on $|l, m\rangle$ with the powers of the creation operators α_{-n} and $\widetilde{\alpha}_{-n}$.

We denote by p_0 and w_0 the operators counting the momentum and the winding number

$$p_0|l, m\rangle = \frac{l}{R}|l, m\rangle, w_0|l, m\rangle = mR|l, m\rangle$$

The operator $\mathrm{e}^{i\frac{l}{R}x_0}$ shifts the momentum. There should also be operators that shift the winding number. We denote them by $\mathrm{e}^{imR\widehat{x}_0}$ so that

$$\mathrm{e}^{i\frac{l_1}{R}x_0}|l, m\rangle = |l + l_1, m\rangle\,, \quad \mathrm{e}^{im_1R\widehat{x}_0}|l, m\rangle = |l, m + m_1\rangle$$

The operators $x_0, p_0, \widehat{x}_0, w_0$ have the commutation relations

$$[x_0, p_0] = i, [\widehat{x}_0, w_0] = i$$

while other commutators vanish.

2 Superstrings

3 String field theory

References