Contents

1 Introduction

These are lecture notes taken from a topics course on Determinants, Analytic Torsion, and Mirror Symmetry taught by Prof. Xianzhe Dai in Winter quarter 2020. These notes were typed up as a collaborative effort by Qingjing Chen, Jiasheng Lin, Danning Lu, Will Sheppard, Chengzhang Sun, Alex Xu, and Junrong Yan.

2 The Determinant of the Laplacian

A reference for this section is *The Laplacian on a Riemannian Manifold*, by Steven Rosenberg. I found a PDF at: http://math.bu.edu/people/sr/articles/book.pdf

2.1 Finite dimensional determinants

Suppose that $A \in M_{n*n}(\mathbb{R})$. Then we know that

$$\det(A) = \prod_{\lambda_i \in \sigma(A)} \lambda_i^{n_i}$$

where $\sigma(A)$ is the spectrum, and n_i is the algebraic multiplicity of λ_i. For the remainder of the discussion in this section, suppose that A is a symmetric, positive definite operator; then

$$\int_{\mathbb{R}^n} e^{-\langle x, Ax\rangle} dx = \pi^{n/2}(\det(A))^{-1/2}$$

This is important to physicists as the partition function. To extend this notion to infinite dimensional operators, we consider the following useful definition.

Definition 2.1. The zeta function corresponding to A is

$$\zeta_A(s) = \sum_{\lambda_i \in \sigma(A)} \lambda_i^{-s}$$

where the eigenvalues λ_i are counted with their corresponding multiplicity. This is a holomorphic function of s; everything is well defined since λ_i are all positive real numbers.

We note that

$$\zeta_A'(s) = -\sum_{\lambda_i \in \sigma(A)} \lambda_i^{-s} \ln \lambda_i$$

In particular, when $s = 0$, we see that

$$\zeta_A'(0) = -\sum_{\lambda_i \in \sigma(A)} \ln \lambda_i = -\ln \det(A)$$

Hence, $e^{-\zeta_A'(0)} = \det(A)$. In the remainder of this section, we will work to extend this definition to the case of infinite dimensional operators by first constructing a "zeta function" in a similar manner. Defined this way, the determinant is usually called the *zeta function regularized determinant*. There are also other ways to "regularize" the determinant which we do not discuss.

2.2 The Hodge Laplacian

Let (M^n, g) be a closed Riemannian manifold, and let $\Omega^*(M) = \Gamma(M, \Lambda^* T^* M)$ denote the space of differential forms over M. Let $d : \Omega^k \to \Omega^{k+1}$ be the de Rham differential. We note that g induces an L^2 inner product on the space of k-forms Ω^k by:

$$\langle \omega, \eta \rangle = \int_M \langle \omega_p, \eta_p \rangle_p d\,\mathrm{vol}_g(p)$$

where $\langle \omega_p, \eta_p \rangle_p$ is the inner product induced on $\Lambda^k T_p^* M$ by letting $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be an orthonormal basis.

Then, d has an adjoint d^* with respect to this inner product. Locally, we can compute an explicit formula for d^* via integration by parts, so d^* takes smooth k-forms to smooth $k-1$-forms.

Now we define $\Delta = dd^* + d^*d : \Omega^* \to \Omega^*$, the Hodge Laplacian. By construction, it is degree preserving. Noting that:

$$\langle \omega, \Delta\omega \rangle = \langle \omega, (dd^* + d^*d)\omega \rangle = \langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle \geq 0,$$

so Δ is positive semidefinite. Furthermore, it can be checked locally that Δ is a second order elliptic operator, and clearly it is also self-adjoint (symmetric); hence, it has discrete real spectrum, and each eigenvalue has finite multiplicity. We list them

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

Moreover, from functional analysis we are indeed sure that $\lambda_k \to +\infty$ as $k \to \infty$ [1].

Remark. Note also that, acting on functions namely 0-forms $\Omega^0(M) = C^\infty(M)$, this Δ is exactly the negative of the classical Laplace-Beltrami operator.

[1] See, for example, Lawrence C. Evans, *Partial Differential Equations*, theorem 6.5.1.

2.3 The determinant of the Laplacian

In analogy with the finite dimensional case, we now define

$$\zeta_\Delta(s) = \sum_{\lambda_i > 0} \lambda_i^{-s}$$

For the moment, none of this is well defined. Mainly we are met with the following:

- when is the above series convergent?
- does $\zeta'_\Delta(0)$ make sense?

It will turn out by understanding asymptotic behavior of the Laplacian that the above sum converges for $\mathrm{Re}(s) > n/2$, with analytic continuation to a meromorphic function on $\mathbb{C}$ with simple poles at $\frac{n}{2} - \ell$, for $\ell \in \mathbb{N}$. That is the goal of this section; we begin motivating discussion with an example:

Example 2.1.1. Let $M = S^1 \subset \mathbb{C}$ be the unit circle with the induced metric. It is not hard to compute:

$$\Delta = -\frac{\partial^2}{\partial \theta^2}$$

on $\Omega^0(M) = C^\infty(M)$. It follows that the eigenvalues are k^2 for $k \in \mathbb{N}$, each with multiplicity 2. The corresponding eigenfunctions are the Fourier basis elements $e^{\pm k\pi i\theta}$. Hence, it follows that

$$\zeta_\Delta(s) = 2\sum_{k=1}^{\infty}(k^2)^{-s} = 2\zeta(2s)$$

Where $\zeta(s)$ is the Riemann zeta function. We recall that the classical Riemann zeta function has an analytic continuation as a meromorphic function on $\mathbb{C}$. In particular, it does not have a pole at $s = 0$. So,

$$\det(\Delta) = \exp(-\zeta'_\Delta(0)) = \exp(-4\zeta'(0)),$$

i.e. the determinant in this case is indeed well-defined. Now we re-examine, in detail, the classical method[2] of analysing the zeta function, namely that which uses *the Gamma function* and *Mellin transform*, in order to be clear on how we could possibly generalize.

Recall the construction of the Gamma function:

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \text{ for } \mathrm{Re}(s) > 0. \tag{1}$$

Which satisfies the functional equation $s\Gamma(s) = \Gamma(s+1)$. From change of variables, for $\lambda > 0$, we obtain $\lambda^{-s} = (1/\Gamma(s))\int_0^\infty t^{s-1}e^{-t\lambda}dt$ (the Mellin transform). So, in our case, we have

$$\zeta_\Delta(s) = 2\sum_{k=1}^{\infty} k^{-2s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(2J(t))dt, \tag{2}$$

where $J(t) = \sum_{k=1}^{\infty} e^{-tk^2}$. Remember that while defined on $\mathrm{Re}(s) > 0$ by the formula (1), the Gamma function extends, via analytic continuation, to a *meromorphic* function on the entire complex plane with simple poles at $s = 0, -1, -2, \ldots$, the non-positive integers. Moreover, from the famous *Euler reflection formula*

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

we see that $\Gamma(s)$ is never zero. Thus the reciprocal $1/\Gamma(s)$ is in fact entire and has zeros at $s = 0, -1, -2, \ldots$.

Now look at the function $J(t)$. It is clear that the series converges pretty well and is in fact smooth on $(0, +\infty)$. Moreover, note that

$$|J(t)| \leq \sum_{k=1}^{\infty} e^{-tk} = \frac{e^{-t}}{1-e^{-t}} \leq \frac{1}{1-1/e}e^{-t}$$

[2]See, for example, Elias Stein and R. Shakarchi, *Complex Analysis*, section 6.2.

for $t \geq 1$. So this tells us that in the part

$$\frac{1}{\Gamma(s)} \int_1^\infty t^{s-1}(2J(t))dt \tag{3}$$

the integral is always absolutely convergent and defines, indeed, an entire function of s. Our problem then remains for the behavior of $J(t)$ as $t \to 0$. For this, we employ the following

Lemma 2.1 (Poisson Summation Formula). *Let $f(t)$ be a smooth real function which decays faster than any inverse power of t as $t \to \pm\infty$. Then*

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{l=-\infty}^{\infty} \widehat{f}(l), \tag{4}$$

where $\widehat{f}(\xi)$ $\left(= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}dt\right)$ is the Fourier transform of $f(t)$.

Thus to the function $f_t(x) = e^{-tx^2}$ whose Fourier transform is $\widehat{f}_t(\xi) = \frac{1}{\sqrt{4\pi t}}e^{-\xi^2/4t}$, this summation formula is readily applied, and we obtain

$$1 + 2\sum_{k=1}^{\infty} e^{-tk^2} = \frac{1}{\sqrt{4\pi t}} + \frac{2}{\sqrt{4\pi t}} \sum_{l=1}^{\infty} e^{-l^2/4t}, \tag{5}$$

implying that, upon noting similarly as before $\sum_{l=1}^{\infty} e^{-l^2/4t} \leq Ce^{-1/4t}$ (now $t \leq 1$),

$$2J(t) = \frac{1}{\sqrt{4\pi t}} - 1 + \mathcal{O}(e^{-1/8t}) \tag{6}$$

as $t \to 0$ (note that $\frac{1}{4t}e^{-1/4t}$ is bounded for $t > 0$). So we write

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}(2J(t))dt &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\left(2J(t) - \frac{1}{\sqrt{4\pi t}} + 1\right) dt + \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{\sqrt{4\pi t}} dt - \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\,(\text{exp. decay})\, dt + \frac{1}{\sqrt{4\pi}\Gamma(s)} \cdot \frac{1}{s - \frac{1}{2}} - \frac{1}{\Gamma(s+1)}, \end{aligned}$$

where in the last term we used the identity $s\Gamma(s) = \Gamma(s+1)$. For the integral in the first term, since we are multiplying a function of exponential decay (as $t \to 0$), it defines an entire function of s. As said previously, the reciprocal of the Gamma function is entire. Therefore, combined with part (3), we conclude that our $\zeta_\Delta(s)$ is indeed analytic in all of $\mathbb{C}$ except the only simple poles at $s = \frac{1}{2}$. In particular, it does not have a pole at 0. Hence, finally, we are allowed to find $\zeta_\Delta'(0)$ which constitutes our definition of the determinant (the Zeta function regularized determinant).

Exercise. Find an explicit formula for $\zeta_\Delta'(0)$. ◀

Now we come to the general case. In the general case, Mellin transform gives

$$\zeta_\Delta(s) = \sum_{\lambda_k > 0} \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\operatorname{Tr}(e^{-t\Delta}) - \dim\ker\Delta\right) dt, \tag{7}$$

where $\operatorname{Tr}(e^{-t\Delta}) = \sum_{\lambda_k \in \sigma(\Delta)} e^{-t\lambda_k}$ counting multiplicity, and thus the number of 1's that appear in the trace is exactly the multiplicity of the zero eigenvalue of Δ, namely $\dim\ker\Delta$; zero eigenvalues are avoided in the sum defining ζ_Δ. Now since the eigenvalues in general can be complicated, tools such as the Poisson summation formula are not available. However, we have the following

Theorem 2.2 (Heat Asymptotic)**.** *Let (M^n, g) be a closed Riemannian manifold, and Δ the Hodge laplacian on $\Omega^i(M)$ (for all i), with its eigenvalues listed, counting multiplicity, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$.*

Then the trace $\mathrm{Tr}(e^{-t\Delta}) = \sum_{k=1}^{\infty} e^{-t\lambda_k}$ *is smooth on* $(0,+\infty)$*, and as* $t \to 0$*, we have a complete asymptotic expansion* $\mathrm{Tr}(e^{-t\Delta}) \sim \sum_{j=0}^{\infty} A_j t^{-\frac{n}{2}+j}$*, that is, for all* $l \in \mathbb{N}$*,*

$$\left| \mathrm{Tr}(e^{-t\Delta}) - \sum_{j=0}^{l} A_j t^{-\frac{n}{2}+j} \right| \leq C_l t^{-\frac{n}{2}+l+1} \tag{8}$$

for $0 < t \leq 1$*, where* C_l *is some constant determined by* l*,*

$$A_0 = \frac{\binom{n}{i} \cdot \mathrm{vol}_g(M)}{(4\pi)^{n/2}}, \text{ and } A_j = \int_M a_j(x) d\,\mathrm{vol}_g, \tag{9}$$

where $a_j(x)$ *is some function depending on the curvature and its derivatives.*

Proof. Coming next. □

As a consequence, we have

Theorem 2.3 (Weyl Asymptotic)**.** *With the same setting as above, for any* $\lambda \geq 0$*, define* $N(\lambda) := \#\{\lambda_i \leq \lambda\}$*, i.e. number of eigenvalues* $\leq \lambda$ *counting multiplicity. Then as* $\lambda \to \infty$*,*

$$N(\lambda) \sim \frac{\binom{n}{i} \cdot \mathrm{vol}_g(M)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)} \lambda^{n/2}. \tag{10}$$

Proof. Coming next. Use the following. □

Lemma 2.4 (Karamata's Tauberian theorem)**.** *Let* $F : [0,+\infty) \to \mathbb{R}$ *be a non-decreasing unbounded function, and let*

$$\omega(t) := \int_0^{\infty} e^{-t\lambda} dF(\lambda) \tag{11}$$

be the Laplace-Stieltjes transform of F*. Then for* $\rho \geq 0$*, we have*

$$\omega(t) \sim Ct^{-\rho} \text{ as } t \to 0 \tag{12}$$

if and only if

$$F(\lambda) \sim \frac{C\lambda^{\rho}}{\Gamma(\rho+1)} \text{ as } \lambda \to \infty. \tag{13}$$

Corollary 2.4.1. *As* $k \to \infty$*,*

$$\lambda_k \sim \left[\frac{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}{\binom{n}{i} \mathrm{vol}_g(M)} k \right]^{2/n}. \quad \square \tag{14}$$

Theorem 2.5 (Zeta function is well-defined)**.** *In the zeta function of our Laplacian*

$$\zeta_{\Delta}(s) = \sum_{\lambda_k > 0} \lambda_k^{-s}, \tag{15}$$

the series is absolutely convergent and defines an analytic function of s *in* $\mathrm{Re}(s) > n/2$*. Moreover, it admits an analytic continuation to all* $s \in \mathbb{C}$ *except the only possible simple poles at* $s = \frac{n}{2}, \frac{n}{2}-1, \ldots, \frac{n}{2}-l, \ldots$*, with* 0 *excluded (i.e. still regular at* 0*).*

Proof. Coming next. Use the following. □

Now we are in a safe position to make

Definition 2.2. Let Δ be our Hodge laplacian. Then define $\det(\Delta) := \exp(-\zeta_{\Delta}'(0))$.

2.4 Asymptotic expansion of heat kernel

Let (M, g) be a closed Riemannian manifold, Δ be the Hodge laplacian on $\Omega^i(M)$. We define

$$Tr(e^{-t\Delta}) := \sum_{i=1}^{\infty} e^{-t\lambda_k},$$

where λ_k are eigenvaules of Δ counting with multiplicities. We want to explore the asymptotic behavior of $Tr(e^{-t\Delta})$ as $t \to 0$.

Let ϕ_k be eigenforms of Δ, i.e. $\Delta\phi_k = \lambda_k\phi_k$. Then we know that $\{\phi_k\}$ forms an orthonormal basis of $L^2(\Omega^i(M))$.

Thus for $f \in L^2(\Omega^i(M))$,

$$f = \sum_{k=1}^{\infty} c_k\phi_k, \text{ where } c_k = (f, \phi_k)_{L^2(\Omega^i(M))} = \int_M < f, \phi_k > dvol_g.$$

Then

$$(e^{-t\Delta} f)(x) = \sum_{k=1}^{\infty} c_k e^{-t\lambda_k}\phi_k = \sum_{k=1}^{\infty} e^{-t\lambda_k}\phi_k(x)\int_M < f, \phi_k > (y)dvol(y).$$

Now define ϕ_k^* by $\phi_k^*(y)(f(y)) =< f(y), \phi_k(y) >$,

$$K(t, x, y) = \sum_{k=1}^{\infty} e^{-t\lambda_k}\phi_k(x) \otimes \phi_k^*(y) \in Hom(\Lambda^i T_y^* M, \Lambda^i T_x^* M).$$

We then have

$$(e^{-t\Delta} f)(x) = \int_M K(t, x, y)f(y)dvol(y).$$

Here $K(t, x, y)$ is called heat kernel of Δ.

So far, we have

- $e^{-t\Delta}$ is an integral operator with integral kernel $K(t, x, y)$.
- $K(t, x, y)$ satisfies the heat equation

$$(\partial_t + \Delta)K(t, x, y) = 0.$$

Moreover, for any $f \in \Omega^i(M)$, $\lim_{t\to 0}\int_M K(t, x, y)f(y)dvol(y) = f(x)$, i.e. $\lim_{t\to 0} K(t, x, y) = \delta_x(y)Id$.

Example 2.2.1. Let $M = (\mathbb{R}^n, g_0)$, where g_0 is the canonical metric on $\mathbb{R}^n$. Then $\Delta = -\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i^2}$ on functions. We can write down the heat kernel explicitly:

$$K_0(t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-d^2(x,y)/t}$$

We now claim $Tr(e^{-t\Delta}) = \int_M tr(K(t, x, x))dvol(x)$. This is because (for simplicity, let's prove the case when Δ acts on functions),

$$K(t, x, x) = \sum_{k=0}^{\infty} e^{-t\lambda_k}|\phi_k(x)|^2.$$

Consequently,

$$\int_M K(t, x, x)dvol(x) = \sum_{k=1}^{\infty} e^{-t\lambda_k} = Tr(e^{-t\Delta}).$$

Since we know that locally, at each $x \in M$, M looks like Euclidean space. So can we approximate $K(t, x, y)$ on functions by $K_0(t, x, y) = \frac{1}{(4\pi t)^{n/2}}e^{-d^2(x,y)/t}$?

In fact, on $B_{\epsilon_0}(x)$, where $\epsilon_0 \leq inj(M)$, we have geodesic coordinates (r, ξ). And

$$\Delta = -\frac{\partial^2}{\partial r^2} - \frac{\partial_r(\sqrt{det(g)})}{\sqrt{det(g)}}\partial_r + \Delta_{S_r^{n-1}},$$

where S_r^{n-1} is the geodesic ball with radius r.

Now set $P_l(t,x,y) = K_0(t,x,y)\sum_{j=0}^{l} t^j u_j(x,y)$, $D = \frac{\sqrt{det(g)}}{2^{n-1}}$ where u_j is to be determined later. By straightforward computation, we know that

$$(\partial_t + \Delta)p_l(t,x,y) = K_0[\frac{r}{2t}\frac{\partial_r D}{D}\sum_{j=0}^{l} t^j u_j + \sum_{j=0}^{l} jt^{j-1}u_j + \sum_{j=0}^{l} t^j \Delta u_j + \frac{r}{t}\sum_{j=0}^{l} t^j \partial_r u_j].$$

If we choose u_j, s.t.

$$\begin{cases} r\partial_r u_0 + \frac{r}{2}\frac{\partial_r D}{D}u_0 = 0, \\ r\partial_r u_j + \frac{r}{2}(\frac{\partial_r D}{D} + j)u_j + \Delta u_{j-1} = 0, \end{cases} \tag{16}$$

then

$$(\partial_t + \Delta)p_l(t,x,y) = t^l(\Delta_y u_l)K_0 = O(t^{l-n/2}).$$

In fact, (16) can be solved recursively. To extend P_l to whole manifold, we choose a smooth cutoff function η, s.t.

$$\eta(x,y) = \begin{cases} 1, & \text{if } d(x,y) \leq \frac{1}{2}inj(M); \\ 0, & \text{if } d(x,y) > inj(M). \end{cases}$$

And set $H_l(t,x,y) = \eta(x,y)P_l(t,x,y)$.

Lemma 2.5.1. *$H_l(t,x,y) \in C^\infty((0,\infty) \times M \times M)$. Moreover,*

1. *if $l > n/2$, $\partial_t + \Delta_y H_l = O(t^{l-n/2})$.*

2. *$H_l(t,x,y) \to \delta_x(y)$ as $t \to 0$.*

In fact, from approximated solution, we have able to derive exact solution(i.e. heat kernel):
Let $F, G \in C^\infty((0,\infty) \times M \times M)$, $F, G = O(1)$ as $t \to 0$. Define

$$(F * G)(t,x,y) := \int_0^t \int_M F(s,x,z)G(t-s,z,ydzds),$$

$$F^{*n} = \underbrace{F * ... * F}_{n \text{ times}}.$$

Then we have the following Duhamel principle

Lemma 2.5.2 (Duhamel Principle)**.** *For fixed $l > n/2$, the heat kernel $K(t,x,y)$ is given by*

$$K = H_l - \left(\sum_{l=1}^{\infty}(-1)^{l+1}\left((\partial_t + \Delta)H_l\right)^{*l}\right) * H_l.$$

As a consequence, we obtain the asyptotic expansion of heat kernel

Theorem 2.6.

$$K(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}}e^{-d^2(x,y)/t}\sum_{j=0}^{\infty} t^j u_j(x,y)$$

as $t \to 0$ for $x, y \in M$ and $d(x,y) \leq \frac{1}{2}inj(M)$.

Moreover, $u_0(x,x) = 1, u_1(x,y) = \frac{1}{3}R(x)$, $u_j(x,x)(j \geq 1)$ depends on the curvature and its derivatives. Here $R(x)$ is the scalar curvature at point x.

Remark. The method extends to the generalized Laplacian: Suppose $E \mapsto M$ be a vector bundle, ∇^E be a connection on E. Then the Bochner's Laplacian is defined

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla_{e_i}e_i},$$

where $\{e_i\}$ are orthonormal basis of TM.

We say L is a generalized Laplacian $L = L^E + F$, for some $F \in C^\infty(M, End(E))$. For example, let $E = \Lambda^i T^*M$, then Hodge Laplacian $\Delta = \Delta^E + \mathcal{R}$ for some curvature term $\mathcal{R}$.

Now suppose K_L is the heat kernel of L, we have

$$K_L(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}} e^{-d^2(x,y)/t} \sum_{j=0}^{l} \Phi_j(x,y),$$

where $\Phi_j(x,j) \in Hom(E_x, E_y)$. Moreover, $\Phi_0(x,y)$ is the parallel transport along radical geodesic with respect to ∇^E.

2.5 Variation of determinant

From our previous discussion, our determinant is clearly depends on metric g. Before moving on, let's look at a concrete example

Example 2.2.2. Let $M = S^1$, then metric on S^1 is sepcific by length L. It's easy to see that

$$\zeta_L(s) = 2(\frac{2\pi}{L})^{-2s}\zeta(2s),$$

which implies

$$\ln det(\Delta_L) = -\zeta_L'(0) = -4\zeta'(0) + 4(\ln(\frac{2\pi}{L}))\zeta'(0).$$

In general, the situation is much more complicated, and local geometry enters.

Recall

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(Tr(e^{-t\Delta}) - B)dt, \text{ where } B = \dim\ker(\Delta).$$

We first study the asymptotic expansion of heat kernel.

Let (M,g) be a closed Riemannian manifold, $E \mapsto M$ be a vector bundle. For a real parameter ϵ, L_ϵ is called a family of generalized laplacian if $L_\epsilon = \Delta^E_\epsilon + F_\epsilon$, where Δ^E_ϵ is the Bochner Laplacian with respect to a C^∞ family of metric g_ϵ and a smooth family of connection $\nabla^{E,\epsilon}$, and F_ϵ is a smooth family in $End(E)$.

Theorem 2.7. *If L_ϵ is a smooth family of generalized Laplacian, then for any $t > 0$, the family of heat kernel $K_\epsilon(t,x,y)$ depends smooth on ϵ (as well as (t,x,y)).*

Moreover,

$$\frac{\partial}{\partial\epsilon}e^{-tL_\epsilon} = -\int_0^t e^{-(t-s)L_\epsilon}\frac{\partial L_\epsilon}{\partial\epsilon}e^{-sL_\epsilon}ds.$$

In particular,

$$\frac{\partial}{\partial\epsilon}Tr(e^{-tL_\epsilon}) = -tTr(\frac{\partial L_\epsilon}{\partial\epsilon}e^{-tL_\epsilon}).$$

Proof. Smooth dependence comes from the construction of heat kernel.

For the derivative, let $\phi \in C^\infty(M,E)$. Then $u_\epsilon := (e^{-tL_\epsilon}\phi)(x) = \int_M K_\epsilon(t,x,y)\phi(y)dy$ solves the initial problem

$$\begin{cases} (\partial_t + L_\epsilon)u_\epsilon = 0, \\ u_\epsilon|_{t=0} = \phi. \end{cases} \tag{17}$$

Differential (17) we get

$$\begin{cases} (\partial_t + L_\epsilon)\frac{\partial u_\epsilon}{\partial\epsilon} = -\frac{\partial L_\epsilon}{\partial\epsilon}u_\epsilon, \\ \partial eu|_{t=0} = 0. \end{cases}$$

As a result,

$$\frac{\partial u_\epsilon}{\partial \epsilon} = \int_0^t e^{-(t-s)L_\epsilon}(-\frac{\partial L_\epsilon}{\partial \epsilon})u_\epsilon(s)ds = \int_0^t e^{-(t-s)L_\epsilon}(-\frac{\partial L_\epsilon}{\partial \epsilon})e^{-sL_\epsilon}\phi ds,$$

which implies

$$\frac{\partial}{\partial \epsilon}Tr(e^{-tL_\epsilon}) = -tTr(\frac{\partial L_\epsilon}{\partial \epsilon}e^{-tL_\epsilon}). \tag{18}$$

□

Now suppose L_ϵ is positive, by (18), we have

$$\frac{\partial}{\partial \epsilon}Tr(e^{-tL_\epsilon}) = t\frac{\partial}{\partial t}Tr(\frac{\partial L_\epsilon}{\partial \epsilon}(L_\epsilon)^{-1}e^{-tL_\epsilon}).$$

Now for $Re(s) > n/2$, by our assumption $B_\epsilon = 0$, hence

$$\zeta_\epsilon(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[Tr(e^{-tL_\epsilon})]dt.$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\zeta_\epsilon(s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\partial}{\partial \epsilon}Tr(e^{-tL_\epsilon})dt \\
&= \frac{1}{\Gamma(s)}\int_0^\infty t^{s}\frac{\partial}{\partial t}Tr(\frac{\partial L_\epsilon}{\partial \epsilon}L_\epsilon^{-1}e^{-tL_\epsilon})dt \\
&= \frac{t^s}{\Gamma(s)}Tr(\frac{\partial L_\epsilon}{\partial \epsilon}L_\epsilon e^{-tL_\epsilon})|_{t=0}^\infty - \frac{1}{\Gamma(s)}\int_0^\infty st^{s-1}Tr(\frac{\partial L_\epsilon}{\partial \epsilon}L_\epsilon^{-1}e^{-tL_\epsilon})dt \\
&= -\frac{1}{\Gamma(s)}\int_0^\infty st^{s-1}Tr(\frac{\partial L_\epsilon}{\partial \epsilon}L_\epsilon^{-1}e^{-tL_\epsilon})dt
\end{aligned}$$

Example 2.2.3. Let (M^2, g) be a closed surface, up to diffeomorphism every metric is conformal to each other, hence it suffices to consider variations of g of the form $g_\epsilon = e^{2\epsilon f}g$ for some $f \in C^\infty(M)$. Then the corresponding variation of Laplacian is given by

$$\triangle_{g_\epsilon} = e^{-2\epsilon f}\triangle_g$$

Hence

$$\frac{\partial \triangle_{g_\epsilon}}{\partial \epsilon} = -2f\triangle_{g_\epsilon}$$

and therefore

$$Tr(\frac{\partial \triangle_{g_\epsilon}}{\partial \epsilon}\triangle_{g_\epsilon}^{-1}e^{-t\triangle_{g_\epsilon}}) = -Tr(2fe^{-t\triangle_{g_\epsilon}})$$

plug this into the formula for $\frac{\partial}{\partial \epsilon}\zeta_{\triangle_{g_\epsilon}}$, we get (drop the integral from 1 to ∞)

$$\frac{\partial}{\partial \epsilon}\zeta_{\triangle_{g_\epsilon}} = -\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(2fe^{-t\triangle_{g_\epsilon}})dt$$

Let $K_\epsilon(t, x, y)$be the heat kernel of $\triangle_{g_\epsilon}$, namely

$$(e^{-t\triangle_{g_\epsilon}}\varphi)(x) = \int_M K_\epsilon(t, x, y)\varphi(y)dy$$

hence

$$(2fe^{-t\triangle_{g_\epsilon}}\varphi)(x) = \int_M 2f(x)K_\epsilon(t, x, y)\varphi(y)dy$$

we treat $2f(x)K_\epsilon(t,x,y)$ as the kernel for the new operator $2fe^{-t\triangle_{g_\epsilon}}$, then the Lidskii theorem says that its trace is given by

$$Tr(2fe^{-t\triangle_{g_\epsilon}}) = \int_M 2f(x)K_\epsilon(t,x,x)dx \tag{19}$$

Now apply the asymptotic expansion for $K_\epsilon(t,x,y)$, we get asymptotic expansion for $2f(x)K_\epsilon(t,x,x)$, which says that

$$2f(x)K_\epsilon(t,x,x) \sim \frac{1}{4\pi t}\sum_{j=0}^{\infty} 2t^j\mu_j(x,x)f(x)$$

here we have $\mu_0(x,x) = 1$ and $\mu_1(x,x) = \frac{1}{3}R$, where R is the scalar curvature. Now if we plug in the asymptotic expansion for $2f(x)K_\epsilon(t,x,x)$ into (19) we can see that

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(2fe^{-t\triangle_{g_\epsilon}})dt$$

has a analytic continuation to $s \in \mathbb{C}$ and is regular at $s = 0$. Moreover we its value at $s = 0$ is given by

$$\left[\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(2fe^{-t\triangle_{g_\epsilon}})dt\right]_{s=0} = \frac{1}{4\pi}\int_M 2f(x)\mu_1(x,x)dx$$

Hence we have:

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\zeta'_{\triangle_{g_\epsilon}}(0) &= \frac{\partial}{\partial s}\Big|_{s=0}\left(\frac{\partial}{\partial\epsilon}\zeta_{\triangle_{g_\epsilon}}(s)\right) \\
&= \frac{\partial}{\partial s}\Big|_{s=0}\left\{s\left[\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(2fe^{-t\triangle_{g_\epsilon}})dt\right]\right\} \\
&= \left[\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr(2fe^{-t\triangle_{g_\epsilon}})dt\right]_{s=0} \\
&= \frac{1}{4\pi}\int_M 2f(x)\mu_1(x,x)dx
\end{aligned}$$

but then we have $\mu_1(x,x) = \frac{1}{3}R_{g_\epsilon} = 2e^{-2\epsilon f}(-\triangle_g(\epsilon f) + K_g)$, and hence

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\ln\det\triangle_{g_\epsilon} &= -\frac{\partial}{\partial\epsilon}\zeta'_{\triangle_{g_\epsilon}}(0) \\
&= -\frac{1}{3\pi}\int_M [\epsilon f(-\triangle_g f) + K_g]\,dvol_g
\end{aligned}$$

If we integrate with respect ϵ from 0 to 1 we get

Theorem 2.8. *(Polyakov formula) If (M^2, g) is a closed surface and $\tilde{g} = e^{2f}g$, then*

$$\ln\det\triangle_{\tilde{g}} - \ln\det\triangle_g = -\frac{1}{6\pi}\int_M (|\nabla f|^2 + 2K_g f)dvol_g$$

In gernal we are not so lucky, although

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr\left(\frac{\partial L_{g_\epsilon}}{\partial\epsilon}L_{g_\epsilon}^{-1}e^{-tL_{g_\epsilon}}\right)dt$$

does admit analytic continuation to $s \in \mathbb{C}$, but $s = 0$ may be a simple pole: to see this we look at the asymptotic expansion of

$$Tr\left(\frac{\partial L_{g_\epsilon}}{\partial\epsilon}L_{g_\epsilon}^{-1}e^{-tL_{g_\epsilon}}\right) = \int_t^\infty Tr\left(\frac{\partial L_{g_\epsilon}}{\partial\epsilon}e^{-sL_{g_\epsilon}}\right)ds$$

and if $K_\epsilon(s,x,y)$ is the integral kernel of L_{g_ϵ} then we have

$$\left(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}e^{-sL_{g_\epsilon}}\varphi\right)(x) = \left(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}\right)_x \int_M K_\epsilon(s,x,y)\varphi(y)dy$$

by the Lidskii theorem again we see that its trace is given by

$$Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}e^{-sL_{g_\epsilon}}) = \int_M \left(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}\right)_x K_\epsilon(s,x,x)dx$$

Recall the asymtotic expansion for $K_\epsilon(t,x,y)$

$$K_\epsilon(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}}e^{-d^2(x,y)/4t}\sum_{j=0}^\infty t^j\mu_j(x,y)$$

which tells us that

$$\left(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}\right)_x K_\epsilon(s,x,y)|_{y=x} \sim t^{-\frac{n}{2}-1}\sum_{j=0}^\infty t^j b_j(x)$$

Thus for $0<t\leq 1$, we have (again drop the integral from 1 to ∞)

$$\begin{aligned}
\int_t^1 Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}e^{-sL_{g_\epsilon}})ds &= \int_t^1 \left[\int_M \left(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}\right)_x K_\epsilon(s,x,x)dx\right]ds \\
&\sim \int_t^1 \left[s^{-\frac{n}{2}-1}\sum_{j=0}^\infty s^j B_j\right]ds \\
&= \sum_{j\neq\frac{n}{2}} \frac{-B_j}{-\frac{n}{2}+j}t^{-\frac{n}{2}+j} + \sum_{j\neq\frac{n}{2}}\frac{-B_j}{-\frac{n}{2}+j} - B_{n/2}\ln t
\end{aligned}$$

where $B_j = \int_M b_j(x)dx$, hence

$$\begin{aligned}
Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}L_{g_\epsilon}^{-1}e^{-tL_{g_\epsilon}}) &= \int_t^\infty Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}e^{-sL_{g_\epsilon}})ds \\
&\sim \sum_{j\neq\frac{n}{2}}\frac{-B_j}{-\frac{n}{2}+j}t^{-\frac{n}{2}+j} + C - B_{n/2}\ln t
\end{aligned}$$

where $C = \sum\limits_{j\neq\frac{n}{2}}\frac{-B_j}{-\frac{n}{2}+j} + \int_1^\infty Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}e^{-sL_{g_\epsilon}})ds$ is a constant.This shows that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr(\frac{\partial L_{g_\epsilon}}{\partial \epsilon}L_{g_\epsilon}^{-1}e^{-tL_{g_\epsilon}})dt$$

does admit an analytic continuation to $s\in\mathbb{C}$, but $s=0$ may be a simple pole due to the term $\ln t$. Of course this will not occur if the dimension of the manifold n is odd, in which case $B_{n/2}=0$.

3 Determinant line bundle

3.1 Introduction

Let $L: V \longrightarrow W$ be a linear map between n dimensional vector spaces, then we can define its determinant as the induced map on the n^{th} exterior power

$$\det L : \bigwedge^n V \longrightarrow \bigwedge^n W$$

equivalently $\det L \in (\bigwedge^n W) \otimes (\bigwedge^n V)^*$ which is a "line", we write this 1 dimensional space as $DetW \otimes (DetV)^*$. Now we want to extend to the infinite dimensional vector spaces, this can be done by exploiting some basic properties of "det":

1. $Det(V \oplus V') \cong (DetV) \otimes (DetV')$

2. If we a short exact sequence of vector spaces:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

then we have $DetV \cong (DetV') \otimes (DetV'')$

3. More generally if we have any linear map $L : V \longrightarrow W$, then we have $(DetW) \otimes (DetV)^* \cong (DetCokerL) \otimes (DetKerL)^*$

The last property enables us to define determinant line for linear maps between vector space not necessarily of finite dimensions, but has finite dimensional kernals and cokernels, in this case we can simply define the determinant line as $(DetCokerL) \otimes (DetKerL)^*$, but a priori this may depend on L.

3.2 Dirac (type) operators

Recall that we have the Hodge laplacian $\triangle = dd^* + d^*d = (d + d^*)^2$ defined on the exterior algebra bundle $\bigwedge^* M$. It is an example of Dirac (type) operator, as one can check that $\triangle = c(e_i)\nabla_{e_i}$ where $\{e_i\}$ is an orthonormal frame of the tangent bundle TM, ∇ being the Levi-Civita connection (here more precisely it's the induced covariant derivative on $\bigwedge^* M$) and $c(e_i)\omega = (e_i^* \wedge \omega) - (e_i \lrcorner \omega)$ is an example of Clifford multiplication, it satisfies the Clifford relation:

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij}$$

Definition 3.0.1. *(Clifford bundle) Let (M^n, g) be a Riemannian manifold, a vector bundle $E \longrightarrow M$ is called a Clifford bundle if there is a bundle homomorphism*

$$\begin{aligned} c \quad : \quad & TM \otimes E \longrightarrow E \\ & (v, s) \longmapsto c(v)s \end{aligned}$$

satisfying

$$c(v)c(w) + c(w)c(v) = -2g(v, w)$$

for any tangent vectors v, w. We call such a c a Clifford mulplication

Example 3.0.1. $E = \bigwedge^* M$ and $c(v) = v^* \wedge -v\lrcorner$ provides an example of Clifford bundle as we have discussed.

Definition 3.0.2. *(Clifford connection) A connection ∇^E on a Clifford bundle is called a Clifford connection if it is compatible with the Clifford connection in the sense that for any vector fields $V, W \in C^\infty(M, TM)$ and a section $s \in C^\infty(M, E)$*

$$\nabla_V^E(c(W)s) = c(\nabla_V W)s + c(W)\nabla_V^E s$$

where ∇ is the usual Levi-Civita connection.

Using the Clifford connection we can define the more general Dirac (type) operator

$$D = c(e_i)\nabla_{e_i}^E : C^\infty(M, E) \longrightarrow C^\infty(M, E)$$

where $\{e_i\}$ is an orthonormal frame of TM. It is known that such a D is a 1^{st} order elliptic differential operator, and is self adjoint with respect to an $L^2 - metric$ $\langle\ ,\ \rangle$ on the completion of the space $C^\infty(M, E)$ provided that $\langle\ ,\ \rangle$ is compatible with the Clifford multiplication in the sense that in the sense that for any unit tangent vector $v \in TM$ we have $\langle c(v)s, c(v)s' \rangle = \langle s, s' \rangle$, such a metric always exists.

Theorem 3.1. *(Lichnerowiz) D^2 is a generalized laplacian, namely*

$$D^2 = \triangle^E + R$$

where $R = \frac{1}{2}c(e_i)c(e_j)R^E(e_i, e_j)$, $\{e_i\}$ is an orthonormal frame and R^E is the curvature os ∇^E.

Proof. $\forall x \in M$, choose a local orthonormal frame $\{e_i\}$ near x such that $\nabla_{e_i} e_j = 0$ at x. Then at x we have

$$\begin{aligned}
D^2 &= (c(e_i)\nabla^E_{e_i})(c(e_j)\nabla^E_{e_j}) \\
&= c(e_i)c(e_j)\nabla^E_{e_i}\nabla^E_{e_j} \\
&= -\nabla^E_{e_i}\nabla^E_{e_i} + \sum_{i\neq j} c(e_i)c(e_j)\nabla^E_{e_i}\nabla^E_{e_j} \\
&= \triangle^E + \frac{1}{2}\sum_{i\neq j} c(e_i)c(e_j)\left[\nabla^E_{e_i}\nabla^E_{e_j} - \nabla^E_{e_j}\nabla^E_{e_i}\right] \\
&= \triangle^E + \frac{1}{2}c(e_i)c(e_j)R^E(e_i, e_j)
\end{aligned}$$

□

Now we would like to discuss a $\mathbb{Z}_2$-grading (supersymmetry) on the Clifford bundle, let's start with an example:

We have the Hodge laplacian $\triangle = D^2$ where $D = d + d^* : \Omega^*(M) \longrightarrow \Omega^*(M)$, we can consider the consider the involution $\sigma = (-1)^{deg} : \bigwedge^* M \longrightarrow \bigwedge^* M$, it gives us a direct sum decomposition $\bigwedge^* M = \bigwedge^{even} M \oplus \bigwedge^{odd} M$, where $\bigwedge^{even} M$ and $\bigwedge^{odd} M$ are eigen-subbundles with eigenvalues 1 and -1 respectively. Moreover it is straightforward to check that σ anticommutes with D, namely $\sigma D = -D\sigma$, hence with respect to the above direct sum decomposition D may be written as

$$D = \begin{bmatrix} 0 & D^{odd} \\ D^{even} & 0 \end{bmatrix}$$

and $(D^{even})^* = D^{odd}$ (because $D^* = D$ is self-adjoint). Such a σ is called a $\mathbb{Z}_2$-grading on $\bigwedge^* M$.

Remark 3.1.1. *In the previous example, we have $ind(D^{even}) := \dim\ker D^{even} - \dim\ker D^{odd} = \chi(M)$ is the Euler characteristic of M.*

Definition 3.1.1. *A $\mathbb{Z}_2$-grading on a Clifford bundle E is a bundle homomorphism $\sigma : E \longrightarrow E$ such that*

$$\begin{cases} \sigma^2 = Id \\ \sigma D + D\sigma = 0 \end{cases}$$

Example 3.0.2. On a complex Clifford bundle over an even dimensional manifold, we can take $\sigma = (-1)^{\frac{n(n+1)}{4}} c(e_1)c(e_2)\cdots c(e_n)$ If n is divisble by 4, then we con't need E to be complex.

Let (E, ∇^E) be Clifford bundle with Clifford connection. Thus we can get a Dirac operator

$$D = c(e_i)\nabla^E_{e_i}.$$

Let $\sigma : E \to E$ be a $\mathbb{Z}_2$-grading, which satisfies

1. $\sigma^2 = \mathrm{Id} \Rightarrow E = E^+ \oplus E^-$.
2. $\sigma D = D\sigma \Rightarrow D = \begin{matrix} o & D^- \\ D^+ & 0 \end{matrix}$.

We have

$$\det D^+ \in (\mathrm{Det}\,\mathrm{Coker}D+) \otimes (\mathrm{Det}\ker D^+)^*$$

3.3 Determinant Line Bundle

Variation of determinant of Laplacian.
Similarily: variation of determinant of Dirac
Geometric description of variation parameters:

$$\begin{tikzcd} M \arrow[r, hook] & X \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Where $M \to X$ is the fiber bundle/fibration, and B is the parameter space.
For $\forall b \in B$, $X_b = \pi^{-1}(b) \cong M$.
$T^V X \subset TX$ is defined to be the tangent vectors of X tangent to the fibers. Assume that we are given the decomposition

$$TX = T^V X \oplus T^H X.$$

Here $T^H X$ is called the horizontal bundle, which is always isomorphic to $\pi^* TB$.
(i.e., the fiber bundle comes with a connection.)
Let g^V be a fiberwise metric on $T^V X$. (family of Riemmannian metrics on typical fiber M.)

Example 3.0.3. Previous, family of generalized Laplacian $\begin{tikzcd} E, \nabla^{E,\epsilon} \arrow[d] \arrow[r, dotted] & M \arrow[r] & X = M \times I \arrow[d, "\pi"] \\ (M, g_\epsilon) & & B = I \end{tikzcd}$ $T^V X = \tilde{\pi}^* TM$, $\tilde{\pi} : M \times I \to M$, $g^V = g_\epsilon$. Given $\begin{tikzcd} M \arrow[r, hook] & X \arrow[d, "\pi"] \\ & B \end{tikzcd}$ with $TX = T^V X \oplus T^H X$, g^V metric on $T^V X$.

Exercise: Choose Riemmanian metric on B, g_B, then $g_X = g^V \oplus \pi^* g_B$ gives Riemmanian metric on X.
Thus we can find connection on $T^V X$ ny projection the Levi-Civita connection:
$\begin{tikzcd} T^V X \arrow[d] \\ X \end{tikzcd}$ Explicitly, if V is a vector field on X, $W \in C^\infty(X, T^V X)$ is a vector field on X tangent to the fibers, then

$$\nabla_V W = (\nabla^L_V W)^V,$$

where ∇^L is the Levi-Civita connection, and for a vector field U on X, U^V is the projection onto the vertical part, given by the decomposition $TX = T^V X \oplus T^H X$.
Exercise: ∇ is independent of the choice of g_B.

Now, $E \to X$ is called the (fiberwise) Clifford bundle, if

$$c : T^V X \otimes E \to E$$

satisfies Clifford relations.
∇^E is called tthe Clifford connection if

$$\nabla^E_V (c(W)s) = c(\nabla_V W)s + c(W)\nabla^E_V s,$$

where $V \in C^\infty(X, TX)$, $W \in C^\infty(X, T^V X)$ and $s \in C^\infty(X, E)$.
and we have family of Dirac operators $\mathcal{D} = c(e_i)\nabla^E_{e_i}$, where $\{e_i\}$ is local orthonormal frame for $T^V X$.
$\forall b \in B$,

$$\mathcal{D}_b : C^\infty(X_b, E|_{E_b}) \to C^\infty(X_b, E|_{E_b})$$

is first order elliptic, self-adjoint (with respect to right metric on E)
Assume additionally, we have a $\mathbb{Z}_2$-grading $\sigma : E \to E$.
We have determinant line

$$(\text{Det Coker}\mathcal{D}_b^+) \otimes (\text{Det ker}\mathcal{D}_b^+)^*.$$

Turns out they patch together to form a smooth line bundle over B, which is called the determinant line bundle. It comes with a natural metric, the Quillen metric, and a compatible connection, the Bismut-Freod connection. The curvature formula for the Bismut-Fried connection elegantly encodes the variation of the determinant of Dirac operators.

Remark. Since $\mathcal{D}^* = \mathcal{D}$, which implies that $(\mathcal{D}^+)^* = \mathcal{D}^-$, which further implies that $\mathrm{Coker}\mathcal{D}^+ \cong \ker\mathcal{D}^-$. Determinant line bundle at $b \in B$ is

$$(\mathrm{Det}\ker\mathcal{D}_b^-) \otimes (\mathrm{Det}\ker\mathcal{D}_b^+).$$

The main issue is that, in general, $\dim\ker\mathcal{D}_b^\pm$ may not be constant in b!

3.4 Quillen's construction of the determinant line bundle

We use the method of Quillen to show that our determinant line bundle is smooth. The main issue really is with the small eigenvalues vanishing; to resolve this, we take them into consideration in this construction. Let

$$\begin{tikzcd} & & E \arrow[dl] \\ M \arrow[r, hook] & X \arrow[d, "\pi"] & \\ & B & \end{tikzcd}$$

be a fiber bundle as before, and let $\mathcal{D}$ be a family of Dirac operators over M. $\sigma : E \to E$ is a $\mathbb{Z}_2$-grading.

Fix $a > 0$, let $U^a = \{b \in B | a \notin \mathrm{Spec}\mathcal{D}_b^2\}$, which is an open subset of B. Let K_b^a be the direct sum of eigenspaces of $\mathcal{D}_b^2$ (which is a generalized Laplacian) with eigenvalue less than a. Thus we have $K^a \to U^a$ as smooth vector bundles. (Remark: only constant dimension over connected parts).Also, $\sigma : K^a \to K^a$ gives decomposition $K^a = K^{a,+} \oplus K^{a,-}$.

Define $\lambda^a = (\mathrm{Det}K^{a,-}) \otimes (\mathrm{Det}K^{a,+})^*$. This is a smooth line bundle over U^a.
Note that we still have short exact sequences

$$0 \to \ker\mathcal{D}^+ \to K^{a,+} \xrightarrow{\mathcal{D}^+} \mathcal{D}^+(K^{a,+}) \to 0,$$

$$0 \to \mathcal{D}^+(K^{a,+}) \to K^{a,-} \to \ker\mathcal{D}^- \to 0.$$

For the second one, note that $\varphi \in K^{a,+}$ is an eigensection implies $\mathcal{D}^-\mathcal{D}^+\varphi = \mathcal{D}^2\varphi = \lambda\varphi$, with $0 \le \lambda < a$; and $\sigma\varphi = \varphi$.
So, we have

$$\lambda^a = (\mathrm{Det}K^{a,-}) \otimes (\mathrm{Det}K^{a,+})^* \cong (\mathrm{Det}\ker\mathcal{D}^-) \otimes (\mathrm{Det}\ker\mathcal{D}^+)^*$$

a smooth line bundle over U^a.

We now describe how the λ^a glue together to form a line bundle; note that since Dirac operators have discrete spectrum, the U^a's indeed form an open cover. Let $0 < a < a'$, and define $U^{a,a'} = U^a \cap U^{a'}$. Define the bundle $K^{a,a'}$ as the direct sum of all eigenspaces corresponding to eigenvalue $a < \lambda < a'$. As before, this is a smooth vector bundle since it has constant rank on each connected component. Notice that

$$K^{a'} = K^a \oplus K^{a,a'}$$

Hence, taking the determinant line bundle we have a canonical isomorphism

$$\lambda^{a'} = \lambda^a \otimes \lambda^{a,a'}$$

Where $\lambda^{a,a'}$ is the determinant line bundle of $K^{a,a'}$. We now note that $\mathcal{D} : K^{a,a'} \to K^{a,a'}$ is a bundle isomorphism, since $K^{a,a'}$ is a direct sum of eigenspaces with bounded below and above eigenvalue. Hence, $\lambda^{a,a'}$ is actually the trivial line bundle so the isomorphism above is actually between $\lambda^{a'}$ and λ^a.

To see that the cocycle condition holds for this system, let $0 < a < a' < a$". We now observe that

$$K^{a,a"} = K^{a,a'} \oplus K^{a',a"}$$

So we get a canonical isomorphism of line bundles

$$\lambda^{a,a"} = \lambda^{a,a'} \otimes \lambda^{a',a"}$$

Which gives us the cocycle condition! Hence, we have shown

Theorem 3.2. *The line bundles λ^a over U^a glue into a line bundle λ defined over all of B. Over each fiber there is a canonical isomorphism $\lambda_b \cong (\mathrm{Det\,Coker}\mathcal{D}_b^+) \otimes (\mathrm{Det\,ker}\mathcal{D}_b^+)^*$.*

Example 3.0.4. Let $M \hookrightarrow X \to B$ be a fiber bundle, and let $E = \Lambda^*(T^V)^*X$ be the Clifford bundle of vertical differential forms. Then $\mathcal{D} = d^V + (d^V)*$ is a Dirac operator, where d^V is the fiberwise differential, and the adjoint is taken with respect to g^V. We take the natural grading given by degree; i.e. $\sigma(\omega) = (-1)^p\omega$ for $\omega \in \Lambda^p$. So even degree forms are the even part of our Clifford bundle and odd degree forms make up the odd part of E.

Then, we note that $\mathcal{D}^2 = \Delta^V$ the fiberwise Hodge-Laplacian. It follows from Hodge theory that $\ker \mathcal{D}^+ \cong H^{even}(M)$, the vector bundle of fiberwise harmonic even forms, and likewise $\ker \mathcal{D}^- \cong H^{odd}(M)$. So in particular, the determinant line bundle $\lambda =$ is already well-defined without any gluing (i.e. there is no weird dimension jumping).

3.5 The Quillem metric on the Determinant Line Bundle

Quillen also showed that the determinant line bundle also carries a natural smooth metric. We begin by motivating this by continuing the example from above.

Example 3.0.5. As before, let E be the clifford bundle of vertical differential forms. There is a fiberwise L^2 metric on the space of sections $\Gamma(X, E)$ given by:

$$\langle \phi, \psi \rangle_b = \int_{X_b} \langle \phi, \psi \rangle dvol_{g_b^V} \in C^\infty(B)$$

This passes down to $\ker \mathcal{D}^\pm$ via identification by fiberwise harmonic forms, and hence a fiberwise metric $\|\cdot\|_{L^2}$ on the determinant line bundle λ. The Quillen metric in this case is

$$\|\cdot\|_Q^2 = \|\cdot\|_{L^2}^2 \cdot \exp \frac{-1}{2}\zeta'_{\mathcal{D}^2}(0)$$

The latter term should be thought of as a correction term; in the general case $\|\cdot\|_{L^2}$ may not be smooth, while $\|\cdot\|_Q$ is!

Now we proceed to the general case. For $a > 0$ fixed, recall that we have a corresponding smooth vector bundle $K^{a,\pm} \to U^a$, endowed with $\|\cdot\|_{L^2,a}$ a fiberwise L^2 metric. As before, this gives rise to a smooth metric on the corresponding determinant bundle λ^a.

We now investigate how the metric changes when we pass from U^a to $U^{a'}$ for $0 < a < a'$. The isomorphism over $U^{a,a'}$ can be described by

$$\begin{gathered}\lambda^a \to \lambda^{a'} \cong \lambda^a \otimes \lambda^{a,a'} \\ s \mapsto s \otimes \det \mathcal{D}^{a,a'}\end{gathered}$$

Note that by self adjointness of $\mathcal{D}$, we have $K^a \perp K^{a,a'}$. Hence,

$$\begin{aligned}\|s \otimes \det \mathcal{D}^{a,a',+}\|_{L^2,a'}^2 &= \|s\|_a^2 \|\det \mathcal{D}^{a,a',+}\|_{L^2,a,a'}^2 \\ &= \|s\|_a^2 (\det(\mathcal{D}^{a,a'})^2)^{1/2}\end{aligned}$$

The second equality comes from self-adjointness of $\mathcal{D}$. So gluing gives a discrepancy, with the correction term coming from the determinant. In light of what we have done with the log determinant, it is not surprising that this should come from:

$$\begin{aligned}\zeta_{\mathcal{D}^2,a}(s) &= \sum_{\lambda > a} \lambda^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[Tr(e^{-\mathcal{D}^2}) - \sum_{\lambda < a} e^{-t\lambda} \right] dt\end{aligned}$$

for $\lambda \in \operatorname{Spec}(\mathcal{D})$. As before, we use zeta function regularization and heat kernel methods to make the integral converge. And so it follows that

$$\| \cdot \|_{Q,a}^2 = \| \cdot \|_{L^2,a}^2 \cdot \exp \frac{-1}{2} \zeta'_{\mathcal{D}^2,a}(0)$$

will agree on the overlaps, and we have proven

Theorem 3.3 (Quillen)**.** *The metric locally defined by $\| \cdot \|_{Q,a}^2$ on U^a is a smooth metric on λ.*

4 Curvature of determinant line bundle

4.1 Torsion of a chain complex

Let $(E, \bar\partial)$ be a chain complex of finite dimensional vector spaces (over $\mathbb{R}$ or $\mathbb{C}$):

$$0 \to E^0 \xrightarrow{\bar\partial} E^1 \xrightarrow{\bar\partial} E^2 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} E^l \to 0, \quad \bar\partial^2 = 0.$$

The *determinant line* of $(E, \bar\partial)$ is defined as

$$\lambda = \operatorname{Det} E := (\operatorname{Det} E^0)^* \otimes (\operatorname{Det} E^1) \otimes (\operatorname{Det} E^2)^* \otimes \cdots ,$$

where the last term is $\operatorname{Det} E^l$ or $(\operatorname{Det} E^l)^*$ depending on the parity of l. Since $\operatorname{Det} E^i$ is one-dimensional, there is a canonical isomorphism $(\operatorname{Det} E^i)^* \otimes (\operatorname{Det} E^i) \cong \mathbb{R}$ or $\mathbb{C}$. It makes sense to denote $(\operatorname{Det} E^i)^* = (\operatorname{Det} E^i)^{-1}$. In this notation,

$$\lambda = \operatorname{Det} E := (\operatorname{Det} E^0)^{-1} \otimes (\operatorname{Det} E^1) \otimes (\operatorname{Det} E^2)^{-1} \otimes \cdots .$$

The *cohomology* of $(E, \bar\partial)$ is $H^i := \ker \bar\partial_i / \operatorname{im} \bar\partial_{i-1}$, where $\bar\partial_i : E^i \to E^{i+1}$. We have short exact sequences

$$0 \to \operatorname{im} \bar\partial_i \to \ker \bar\partial_{i+1} \to H^{i+1} \to 0,$$

$$0 \to \ker \bar\partial_{i+1} \to E^{i+1} \to \operatorname{im} \bar\partial_{i+1} \to 0.$$

Hence, $\operatorname{Det} E^{i+1} \cong \operatorname{Det} H^{i+1} \oplus \operatorname{Det} \operatorname{im} \bar\partial_i \oplus \operatorname{Det} \operatorname{im} \bar\partial_{i+1}$, and thus we have a canonical isomorphism

$$\lambda = \operatorname{Det} E \cong \operatorname{Det} H^* := (\operatorname{Det} H^0)^{-1} \otimes (\operatorname{Det} H^1) \otimes \cdots . \tag{20}$$

Remark. In algebraic topology, we have an analogue (Euler-Poincaré formula)

$$\sum_{i=0}^{l} (-1)^i \dim E^i = \sum_{i=0}^{l} (-1)^i \dim H^i.$$

Now assume that $(E, \bar\partial)$ is acyclic, i.e., $H^i(E) = 0$ for all i. Then the canonical isomorphism (20) becomes

$$\lambda = \operatorname{Det} E \cong \mathbb{R} \text{ or } \mathbb{C},$$

i.e., there is a canonical nonzero vector $T(\bar\partial) \in \lambda$, called the *torsion* of $(E, \bar\partial)$.

In fact, the torsion can be constructed as follows. As the complex is acyclic, the pair of short exact sequences reduces to

$$0 \to \operatorname{im} \bar{\partial}_i \to E^{i+1} \to \operatorname{im} \bar{\partial}_{i+1} \to 0.$$

Let $n_i = \dim \operatorname{im} \bar{\partial}_i$, and $\bar{v}_1^{(i+1)}, \dots, \bar{v}_{n_i}^{(i+1)}$ form a basis of $\operatorname{im} \bar{\partial}_i \subseteq E^{i+1}$. Then for $j = 1, \dots, n_i$, there exists $v_j^{(i)} \in E^i$ such that $\bar{v}_j^{(i+1)} = \bar{\partial} v_j^{(i)}$. Let

$$s^i = v_1^{(i)} \wedge \cdots \wedge v_{n_i}^{(i)} \in \bigwedge^{n_i} E^i.$$

Then

$$\bar{\partial} s^i = \bar{\partial} v_1^{(i)} \wedge \cdots \wedge \bar{\partial} v_{n_i}^{(i)} = \bar{v}_1^{(i+1)} \wedge \cdots \wedge v_{n_i}^{(i+1)} \in \bigwedge^{n_i} E^{i+1},$$

and

$$\bar{\partial} s^i \wedge s^{i+1} = \bar{v}_1^{(i+1)} \wedge \cdots \wedge v_{n_i}^{(i+1)} \wedge v_1^{(i+1)} \wedge \cdots \wedge v_{n_{i+1}}^{(i+1)} \in \operatorname{Det} E^{i+1}.$$

It is nonzero because $\bar{v}_1^{(i+1)}, \dots, \bar{v}_{n_i}^{(i+1)}, v_1^{(i+1)}, \dots, v_{n_{i+1}}^{(i+1)}$ form a basis of E^{i+1}.

Definition 4.1. The torsion of the acyclic chain complex $(E, \bar{\partial})$ is

$$T(\bar{\partial}) = (s^0)^{-1} \otimes (\bar{\partial} s^0 \wedge s^1) \otimes (\bar{\partial} s^1 \wedge s^2)^{-1} \otimes \cdots \in \lambda \setminus \{0\}.$$

Clearly this is independent of the choice of bases.

Example 4.1.1. Any short exact sequence

$$0 \to E^0 \xrightarrow{i} E^1 \xrightarrow{j} E^2 \to 0$$

is an acyclic chain complex. In this case the torsion gives a canonical isomorphism

$$\lambda = \operatorname{Det} E = (\operatorname{Det} E^0)^{-1} \otimes (\operatorname{Det} E^1) \otimes (\operatorname{Det} E^2)^{-1} \cong \mathbb{R} \text{ or } \mathbb{C},$$

i.e.,

$$\operatorname{Det} E^1 \cong (\operatorname{Det} E^0) \otimes (\operatorname{Det} E^2),$$

which has been used repeatedly.

Now let $(E, \bar{\partial})$ be endowed with an inner product (or Hermitian inner product). Then the determinant line inherits an inner product.

Definition 4.2. The analytic torsion of the acyclic chain complex (E, δ) is

$$\tau(\bar{\partial}) = |T(\bar{\partial})|.$$

Since $\bar{\partial}_i : E^i \to E^{i+1}$, its dual map is $\bar{\partial}_i^* : E^{i+1} \to E^i$. Put

$$D = \bar{\partial} + \bar{\partial}^* : \bigoplus_{i=0}^{l} E^i \to \bigoplus_{i=0}^{l} E^i.$$

Then

$$D^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

preserves the degree. Denote $D_i^2 = D^2|_{E^i} : E^i \to E^i$. Then $\det D_i^2 \in \mathbb{R}^+$.

Proposition 4.1. $\ln \tau(\bar{\partial}) = \frac{1}{2} \sum_{i=0}^{l} (-1)^{i+1} i \ln \det D_i^2$.

Proof. First, note that $E^i = \bigoplus_{\lambda \in \operatorname{Spec} D_i^2} E^i(\lambda)$, where $E^i(\lambda)$ is the eigenspace of D_i^2 with eigenvalue λ. Since $\bar{\partial} D^2 = D^2 \bar{\partial}$, we have $\bar{\partial} : E^i(\lambda) \to E^{i+1}(\lambda)$ and thus

$$(E, \bar{\partial}) = \bigoplus_{\lambda \in \operatorname{Spec} D^2} (E(\lambda), \bar{\partial}).$$

Hence without loss of generality, we can assume D^2 has only one eigenvalue $\lambda > 0$. [3]

Secondly, if $v \in E^i$, then $D^2 v = \lambda v$, and thus

$$v = \frac{1}{\lambda} D^2 v = \frac{1}{\lambda}\bar\partial\bar\partial^* v + \frac{1}{\lambda}\bar\partial^*\bar\partial v =: v_1 + v_2,$$

where $v_1 \in \operatorname{im}\bar\partial$, $v_2 \in \operatorname{im}\bar\partial^*$. We have an orthogonal decomposition

$$E^i = \operatorname{im}\bar\partial_{i-1} + \operatorname{im}\bar\partial^*_{i+1}.$$

Proof: $\langle \partial u_1, \partial^* u_2\rangle = \langle \partial^2 u_1, u_2\rangle = 0$.

Now we choose an orthonormal basis $v_1^{(i)}, \dots, v_{n_i}^{(i)}$ for $\operatorname{im}\bar\partial_i$, and then $\lambda^{-1/2}\bar\partial v_1^{(i)}, \dots, \lambda^{-1/2}\bar\partial v_{n_i}^{(i)}$ form an orthonormal basis of $\operatorname{im}\bar\partial_i$. Proof: $\langle \lambda^{-1/2}\bar\partial v_j^{(i)}, \lambda^{-1/2}\bar\partial v_k^{(i)}\rangle = \lambda^{-1}\langle \bar\partial^*\bar\partial v_j^{(i)}, v_k^{(i)}\rangle = \lambda^{-1}\langle D^2 v_j^{(i)}, v_k^{(i)}\rangle = \langle v_j^{(i)}, v_k^{(i)}\rangle = \delta_{jk}$, and $\bar\partial : \operatorname{im}\bar\partial^*_{i+1} \to \operatorname{im}\bar\partial_i$ is an isomorphism. Hence,

$$T(\bar\partial) = (v_1^{(0)} \wedge \cdots \wedge v_{n_0}^{(0)}) \otimes (\bar\partial v_1^{(0)} \wedge \cdots \wedge \bar\partial v_{n_0}^{(0)} \wedge v_1^{(1)} \wedge \cdots \wedge v_{n_1}^{(1)}) \otimes \cdots.$$

However, $|\lambda^{-1/2}\bar\partial v_1^{(i)} \wedge \cdots \wedge \lambda^{-1/2}\bar\partial v_{n_i}^{(i)} \wedge v_1^{(i+1)} \wedge \cdots \wedge v_{n_{i+1}}^{(i+1)}| = 1$, where $n_i = \dim\operatorname{im}\bar\partial^*_{i+1} = \dim\operatorname{im}\bar\partial_i$. Hence, $|\bar\partial v_1^{(i)} \cdots \wedge \bar\partial v_{n_i}^{(i)} \wedge v_1^{(i+1)} \wedge \cdots \wedge v_{n_{i+1}}^{(i+1)}| = \lambda^{n_i/2}$. Thus we have

$$\tau(\bar\partial) = |T(\bar\partial)| = \lambda^{\frac{1}{2}\sum_{i=0}^{l}(-1)^i n_i}.$$

And $\det D_i^2 = \lambda^{\dim E^i} = \lambda^{n_i + n_{i-1}}$. Since

$$\begin{aligned}\sum_{i=0}^{l}(-1)^{i+1} i(n_i + n_{i-1}) &= \sum_{i=0}^{l}(-1)^{i+1} i n_i - \sum_{i=0}^{l-1}(-1)^i (i+1) n_i \\ &= \sum_{i=0}^{l}(-1)^i n_i, \qquad (n_l = 0)\end{aligned}$$

this completes the proof. □

Remark. 1. Both $T(\bar\partial)$ and $\tau(\bar\partial)$ can be defined in general, but will depend on the choice of volume forms on H^i.

2. Under canonical isomorphism, $T(\bar\partial) = \det D^+$ up to scaling.

4.2 Holomorphic vector bundles

Let M be a *complex manifold*, i.e., there is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ where $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ is such that

$$\varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic.

Example 4.2.1. $\mathbb{CP}^n$ is a complex manifold. The open subset $GL(k, mathbbC) \subseteq \mathbb{C}^{k^2}$ is also a complex manifold.

Let $M = \mathbb{C}$ and $z = x + iy \in M$. We define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then $f \in C^\infty(M)$ is holomorphic if and only if $\frac{\partial f}{\partial \bar z} = 0$ (Cauchy-Riemann equation). Define $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\bar z = \mathrm{d}x - i\,\mathrm{d}y$. Then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = \frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar z}\,\mathrm{d}\bar z =: \partial f + \bar\partial f.$$

[3] Here we are using the fact that $\tau(\bar\partial \oplus \bar\partial') = \tau(\bar\partial)\tau(\bar\partial')$ and that $\det(D_i \oplus D_i')^2 = \det D_i^2 \det D_i'^2$.

We have a decomposition $d = \partial + \bar\partial$.

More generally, $M = \mathbb{C}^n$, and $(z_1, \dots, z_n) \in M$. Write $z_i = x_i + \sqrt{-1}y_i$. Define

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right);$$

$$\mathrm{d}z_i = \mathrm{d}x_i + \sqrt{-1}\,\mathrm{d}y_i, \quad \mathrm{d}\bar z_i = \mathrm{d}x_i - \sqrt{-1}\,\mathrm{d}y_i;$$

$$\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z}\,\mathrm{d}z, \quad \bar\partial f = \sum_{i=1}^n \frac{\partial f}{\partial \bar z}\,\mathrm{d}\bar z.$$

Then $TM \otimes \mathbb{C}$ has a decomposition

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M,$$

where $T^{(1,0)}M$ is spanned by $\frac{\partial}{\partial z_i}$, $i = 1, \dots, n$, and $T^{(0,1)}M$ is spanned by $\frac{\partial}{\partial \bar z_i}$, $i = 1, \dots, n$.

This holds in general for any complex manifold M.

- $TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$, $T^*M \otimes \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M$, and $d = \partial + \bar\partial$.
- f is holomorphic if and only if $\bar\partial f = 0$.
- $\bigwedge^k(T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \bigwedge^{p,q} M$, where $\bigwedge^{p,q} M = \bigwedge^p(T^{*(1,0)}M) \otimes \bigwedge^q(T^{*(0,1)}M)$. $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$.
- $d = \partial + \bar\partial$ extends to forms:

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

  And $d^2 = 0$ implies $\partial^2 = \bar\partial^2 = 0$, $\partial\bar\partial + \bar\partial\partial = 0$.

A *smooth* $\mathbb{C}$*-vector bundle* $\pi : E \to M$ has local trivialization

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k,$$

such that the transition map

$$\phi_{\alpha\beta} = \phi_\alpha\phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^k \to (U_\alpha \cap U_\beta) \times \mathbb{C}^k$$

is given by $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ is smooth. We say that π is *holomorphic* if $g_{\alpha\beta}$ is holomorphic.

Example 4.2.2. Let $M = \mathbb{CP}^n$, with homogeneous coordinates $[z_0, \dots, z_n]$. Let $U_i = \{z_i \neq 0\}$ and the local chart is $\varphi_\alpha : U_i \to \mathbb{C}^n$,

$$[z_0, \dots, z_n] \mapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right).$$

Then $\varphi_{ij} = \varphi_i\varphi_j^{-1} : \varphi^j(U_i \cap U_j) \to \varphi^i(U_i \cap U_j)$ is holomorphic. Consider the trivial line bundle $U_i \cap \mathbb{C}$. We glue them together by

$$g_{ij} : U_i \cap U_j \to GL(k, \mathbb{C}), \quad [z_0, \dots, z_n] \mapsto \left(\frac{z_i}{z_j}\right)^k, \quad k \in \mathbb{Z}.$$

Clearly the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ is satisfied and thus we have a homolomorphic line bundle

$$L_k \to \mathbb{CP}^n,$$

which is often denoted by $\mathcal{O}(k)$.

Example 4.2.3. For any complex manifold M, $T^{(1,0)}M$, $T^{*(1,0)}M$ are holomorphic vector bundles over M. ($\bar\partial g_{ij} = 0 \implies \bar\partial(\partial g_{ij}) = 0$.)

Let $\pi : E \to M$ be a holomorphic vector bundle. Then we have a local frame $e_1, \ldots, e_k$ which is "holomorphic" in the sense that if $e'_1, \ldots, e'_k$ is another such frame, then $e'_i = \sum a_{ij} e_j$ where a_{ij}'s are holomorphic. Hence, if $s \in C^\infty(M, E)$ is a smooth section, then $s = s^i e_i$ and we can define

$$\bar{\partial}_E s = \bar{\partial} s^i \otimes e_i \in C^\infty(M, T^{*(0,1)}M \otimes E).$$

The operator $\bar{\partial}_E : \Omega^{(0,0)}(M, E) \to \Omega^{(0,1)}(M, E)$ is called the *Dolbeault operator*.

Example 4.2.4. If $e_1, \ldots, e_k$ is a holomorphic local frame, then $\bar{\partial}_E e_i = 0$.

Like the exterior differential d, the Dolbeault operator extends to

$$\bar{\partial}_E : \Omega^{(0,q)}(M, E) \to \Omega^{(0,q+1)}(M, E)$$

using Leibniz rule

$$\bar{\partial}_E(\omega \otimes s) = (\bar{\partial}\omega) \otimes s + (-1)^{\deg \omega} \omega \otimes \bar{\partial}_E s.$$

We get the *Dolbeault complex*

$$0 \to \Omega^{(0,0)}(M, E) \xrightarrow{\bar{\partial}_E} \Omega^{(0,1)}(M, E) \xrightarrow{\bar{\partial}_E} \cdots \xrightarrow{\bar{\partial}_E} \Omega^{(0,n)}(M, E) \to 0, \quad \bar{\partial}_E^2 = 0.$$

The *Dolbeault cohomology* is

$$H^{0,q}(M, E) = \frac{\ker \bar{\partial}_E|_{\Omega^{0,q}}}{\operatorname{im} \bar{\partial}_E|_{\Omega^{0,q-1}}}.$$

Example 4.2.5. $H^{0,0}(M, E)$ is the set of global sections of $\pi : E \to M$.

Remark. Take $E = \bigwedge^p (T^{*(1,0)}M)$. Then $\Omega^{0,q}(M, E) = \Omega^{p,q}(M)$.

Recall that a connection on $\pi : E \to M$ is a map

$$\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes_{\mathbb{R}} E).$$

Since $E = \mathbb{C} \otimes_{\mathbb{C}} E$, we have $T^*M \otimes_{\mathbb{R}} E = (T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E$. Hence, there is a decomposition

$$\nabla = \nabla' + \nabla'', \quad \nabla' : C^\infty(M, E) \to C^\infty(M, T^{*(1,0)}M \otimes E), \quad \nabla'' : C^\infty(M, E) \to C^\infty(M, T^{*(0,1)}M \otimes E).$$

We say that ∇ is compatible with the holomorphic structure if $\nabla'' = \bar{\partial}_E$.

Definition 4.3. A connection ∇ on a vector bundle (not necessarily holomorphic) E also induces a connection 1-form $\omega \in \Omega^1(M, \operatorname{End}(E))$. To define this, let $e_1, \cdots, e_k$ be a local frame, and let $\omega_i^j \in \Omega^1(M)$ such that

$$\nabla e_i = \sum_{j=1}^k \omega_i^j e_j$$

In fact, it is not hard to show via Leibniz rule that $\nabla = d + \omega$. Conversely, $\omega \in \Omega^1(M, \operatorname{End}(E))$ induces a connection via $\nabla = d + \omega$.

Theorem 4.2 (Chern connection)**.** *Every holomorphic vector bundle $\pi : E \to M$ with a hermitian metric admits a unique connection (called the* Chern connection*) compatible with both the holomorphic structure and the hermitian metric, i.e.,*

1. $\nabla'' = \bar{\partial}_E$,
2. $\mathrm{d}\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle$, *for any* $s, t \in C^\infty(M, E)$.

Proof. Uniqueness. Let (e_i) be a local holomorphic frame, $h_{ij} = \langle e_i, e_j\rangle$ and $H = (h_{ij})$. Then

$$\mathrm{d}h_{ij} = \langle \nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle.$$

Since

$$\nabla e_i = \nabla' e_i + \nabla'' e_i = \nabla' e_i + \bar{\partial}_E e_i = \nabla' e_i = \omega_i^j e_j$$

where ω_i^j's are $(1,0)$-form (connection form), we have

$$\mathrm{d}h_{ij} = \partial h_{ij} + \bar{\partial} h_{ij} = \omega_i^k h_{kj} + \bar{\omega}_j^k h_{ik} \implies \partial h_{ij} = \omega_i^k h_{kj}, \quad \bar{\partial} h_{ij} = \bar{\omega}_k^k h_{ik}.$$

Therefore, $(\omega_i^j) = \partial H \cdot H^{-1}$ is uniquely determined.

Existence. Define $\omega = (\omega_i^j) = \partial H \cdot H^{-1}$. Then one can check that $\nabla = d + \omega$ is a Chern connection. □

In particular, the connection 1-form of a Chern connection is $(\omega_i^j) = \partial H \cdot H^{-1}$ (as outlined in the above proof).

Definition 4.4. The **curvature** of a connection ∇ is a 2-form defined by $\Omega = \nabla \circ \nabla$. This can be checked to be C^∞-linear (and thus a tensor), and in fact $\Omega = d\omega + \omega \wedge \omega$.

Example 4.4.1. Suppose that $L \to M$ is a holomorphic line bundle with hermitian metric and a locally nonvanishing holomorphic section e. Then $h = \langle e, e \rangle$ and the connection 1-form associated to the Chern connection is

$$\omega = (\partial h) \cdot h^{-1} = \frac{\partial h}{h} = \partial \ln h$$

Since L is a line bundle, the $\omega \wedge \omega$ part of the curvature tensor vanishes. Hence, the curvature will be

$$\Omega = d\omega = (\partial + \overline{\partial}) \partial \ln h = \overline{\partial} \partial \ln h$$

In particular, the curvature is a (1,1) form $\Omega \in \Omega^{(1,1)}(L)$.

Remark. This is a feature of Chern connections. I.e. a connection over a hermitian vector bundle is Chern if and only if the connection is a (1,1) form.

For the rest of this section, whenever we talk about the curvature of a hermitian vector bundle, we mean with respect to the Chern connection.

4.3 Holomorphic Determinant Line Bundles

Let B be a complex manifold, $dim_{\mathbb{C}} B = m$. For $i = 0, 1, \cdots, \ell$, let $E_i \to B$ be a sequence of finite rank holomorphic vector bundles, with a chain complex structure:

$$0 \to E_0 \xrightarrow{v} E_1 \xrightarrow{v} \cdots \xrightarrow{v} E_\ell \to 0$$

i.e. v are holomorphic vector bundle homomorphisms ($\overline{\partial} v = 0$), with $v^2 = 0$.

Definition 4.5. The above defines a **holomorphic chain complex**, denoted (E, v)

If we endow the E_i's with hermitian metrics, then the corresponding determinant bundle

$$\lambda = (\det E_0)^{-1} \otimes (\operatorname{Det} E_1) \otimes (\det E_2)^{-1} \otimes \cdots$$

will be a holomorphic line bundle endowed with a natural hermitian metric. To do this, we can find a local holomorphic frame from the construction in section 4.1. Then it follows that there is a metric on each determinant bundle that we may then multiply together to get the metric on λ.

Furthermore, if (E, v) is acyclic, then we may find a canonical nonvanishing section of λ via the torsion $T(v) \in \Gamma(\lambda)$. The curvature of λ is thus

$$-\partial \overline{\partial} \ln |T(v)|^2 = -2 \partial \overline{\partial} \ln \tau(v)$$

However, we may also find the curvature of λ by first considering the curvatures of each E_i. Explicitly, let Ω_i be the curvature of E_i. Then we may show that the curvature of λ is

$$\sum_{i=0}^{\ell} (-1)^{i+1} tr(\Omega_i)$$

by showing that

1. The curvature of $\det E_i$ is $\Omega_{\det E} = tr(\Omega_i)$
2. The curvature of a tensor product $E \otimes E'$ is $\Omega_{E\otimes E'} = \Omega_E \otimes 1_{E'} + 1_E \otimes \Omega_{E'}$
3. The curvature of the dual bundle of a line bundle L^{-1} is negative of the curvature of L

Then it follows that we have the equality

$$\sum_{i=0}^{\ell} (-1)^{i+1} tr(\Omega_i) = -2\partial\overline{\partial} \ln \tau(v)$$

We now proceed to discuss another perspective that will generalize well in the infinite dimensional case. Write $E = \bigoplus E_i$.

Definition 4.6. (Number operator) Let $N : E \to E$ by $N(s) = j \cdot s$, for $s \in E_j$. (So it multiplies elements of E_j by the integer j)

Definition 4.7. (Sign operator) Let $\sigma : E \to E$ by $N(s) = (-1)^j \cdot s$, for $s \in E_j$. This will split E into even and odd parts via +1 and -1 eigenspace decomposition (i.e. a $\mathbb{Z}_2$-grading). So here we have $E = E^+ \oplus E^-$ by $E^+ = E_0 \oplus E_2 \oplus \cdots$ and $E^- = E_1 \oplus E_3 \oplus \cdots$.

We also write $v^* : E_j \to E_{j+1}$ to be the adjoint of v with respect to the hermitian metric on E. This gives rise to the corresponding Dirac operator $V = v + v^*$. Noting that V reverses parity (i.e. $V : E^{\pm} \to E^{\mp}$), we may thus write it in block antidiagonal form with respect to the decomposition $E = E^+ \oplus E^-$

$$V = \begin{pmatrix} 0 & v^- \\ v^+ & 0 \end{pmatrix}$$

In addition, the $\mathbb{Z}_2$ grading also gives us a supertrace. For $A \in \Gamma(M, End(E))$, this is defined by

$$Tr_s(A) = tr(\sigma \circ A)$$

We may also extend this to $\Omega^*(M, End(E))$ by $Tr_s(\omega A) = \omega Tr_s(A)$.

Now let $\nabla = \bigoplus \nabla^{(i)}$ be the Chern connection over E. For $u > 0$, we define

$$\mathbb{A}_u = \nabla + \sqrt{u} V$$

This is the first example of a **superconnection**; the discussion here is unfortunately quite vague, for more details on the definition of superconnection, read "Heat Kernels and Dirac Operators" by Berline-Getzler-Vergne.

Note that the curvature will, as before, be an element of $\Omega^*(M, End(E))$ (i.e. no differentiation occurs). To see this, first

$$\mathbb{A}_u^2 = \nabla^2 + \sqrt{u}(V\nabla + \nabla V) + uV^2$$

The proof will thus follow if we can show $(V\nabla + \nabla V)$ is 0th order. This follows from the Bianchi identities (?).

We now define a fiberwise zeta function arising from E:

$$\zeta_E(s) = \frac{-1}{\Gamma(s)} \int_0^{\infty} u^{s-1} Tr_S(N \cdot \exp -\mathbb{A}_u^2) du$$

Here $s \in \mathbb{C}$ and Γ is the gamma function.

Proposition 4.3. *$\zeta_E(s)$ is holomorphic for $Re(s) > 0$. Furthermore, it analytically extends to a holomorphic function over $\mathbb{C}$*

Do note that this is a ζ-function defined over every point $b \in B$, and that it outputs E-valued differential forms. The proof of analyticity is trivial because $\mathbb{A}_u$ is a finite dimensional operator.

Example 4.7.1. The degree 0 part of $\zeta_E(s)$ is

$$[\zeta_E(s)]_0 = \frac{-1}{\Gamma(s)}\int_0^\infty u^{s-1}Tr_S(N\cdot\exp -uV^2)du$$

Since V^2 does not change the differential grading.

In fact for acyclic chain complexes, there is more to say

Theorem 4.4. *If the chain complex (E, v) is acyclic, then*

$$Tr_s(\exp -\nabla^2) = \partial\overline{\partial}\zeta_E'(0)$$

For example, looking at the degree 2 part of the equation, we have that

$$Tr_s(-\nabla^2) = \partial\overline{\partial}[\zeta_E'(0)]_0$$

Proof. The key to this lies in the transgression formula:

$$\frac{\partial}{\partial u}Tr_s(\exp\mathbb{A}_u^2) = \frac{1}{u}\partial\overline{\partial}Tr_s(N\cdot\exp -\mathbb{A}_u^2)du$$

(To be proven at a later date). Assuming that the formula is true, the theorem then follows via integration of the above identity

$$Tr_s(\exp\mathbb{A}_u^2)\Big|_{u=0}^{\infty} = \int_0^\infty \frac{1}{u}\partial\overline{\partial}Tr_s(N\cdot\exp -\mathbb{A}_u^2)du$$

Now when (E, v) is acyclic V is invertible (from a Hodge decomposition). Computing, the LHS will thus be

$$Tr_s(\exp\mathbb{A}_u^2)\Big|_{u=0}^{\infty} = -Tr_s(\exp -\nabla^2)$$

and the RHS can be computed via first noting that

$$\begin{aligned}\zeta_E'(0) = &-\int_0^1 u^{-1}[Tr_S(N\cdot\exp -\mathbb{A}_u^2) - Tr_s(N\cdot\exp -\nabla^2)]du \\ &+\int_1^\infty u^{-1}Tr_S(N\cdot\exp -\mathbb{A}_u^2)du + \Gamma(1)Tr_s(N\cdot\exp -\nabla^2)\end{aligned}$$

Then from Chern-Weil theory, □

4.4 Aside on Chern-Weil Theory

Roughly speaking, the Chern-Weil theory can be seen as a "geometric" theory of characteristic classes, which relates the local geometric information (such as curvature) to the global topological properties of the vector bundle.

Let M be a real smooth manifold, $E \to M$ a vector bundle, and ∇ a connection on E; that is, a (real) linear map $C^\infty(M, E) \to \Omega^1(M, E)$ satisfying $\nabla(f\cdot s) = (df)\otimes s + f\cdot\nabla s$ for all $s \in C^\infty(M, E)$ and $f \in C^\infty(M)$. Note that the domain of definition of ∇ can be extended over all (E-valued) forms $\Omega^*(M, E)$. Thus from this connection we have also defined its *curvature* $\Omega = \nabla^2$. The curvature can also be seen as a element of $\Omega^2(M, \mathrm{End}(E))$.

Example 4.7.2. Let $E = M\times\mathbb{R}^n$ be the trivial bundle, then just setting $\nabla = d$, the exterior differential, defines a connection. This connection has zero curvature since $\Omega = d^2 = 0$. More generally, pick $A \in \Omega^1(M, M_k(\mathbb{R}))$, then $\nabla = d + A$ also defines a connection, with its curvature $\Omega = dA + A\wedge A$.

Remark. It can be shown that *any* connection on $E \to M$ will look like $d + A$ locally (or equivalently, on a trivial bundle).

Remark. Given connection ∇ and a vector field X on M, we may obtain operators ∇_X by substituting X into the form obtained as the value of ∇. In this way, it can be shown that our curvature has the usual expression

$$\Omega(X,Y)s = \nabla_X\nabla_Y(s) - \nabla_Y\nabla_X(s) - \nabla_{[X,Y]}(s), \tag{21}$$

for vector fields X, Y and bundle section s.

We have also defined the trace functional $\mathrm{Tr} : \Omega^*(M, \mathrm{End}(E)) \to \Omega^*(M)$ for which $\mathrm{Tr}(\omega \otimes A) := \omega\,\mathrm{Tr}(A)$ for $\omega \in \Omega^*(M)$ and $A \in C^\infty(M, \mathrm{End}(E))$. Abbreviate $\omega \otimes A$ by ωA, we define $[\omega A, \eta B] := \omega A \wedge \eta B - (-1)^{\deg\omega\cdot\deg\eta}(\eta B) \wedge (\omega A)$. From this definition we easily see that $\mathrm{Tr}([\omega A, \eta B]) = 0$.

Example 4.7.3. Let ∇ be a connection on E and $A \in C^\infty(M, \mathrm{End}(E))$. For the curvature $\Omega = \nabla^2 \in \Omega^2(M, \mathrm{End}(E))$, we have $[\Omega, A] = \Omega A - A\Omega$ (note ΩA means "matrix multiplication", noting that $\mathrm{End}(E)$ is in fact a bundle of algebras??); so we can check that $\mathrm{Tr}([\Omega, A]) = 0$.

Lemma 4.5. *Let ∇ be a connection on E, then for any $A \in \Omega^*(M, \mathrm{End}(E))$, we have*

$$[\nabla, A] = \nabla \circ A - (-1)^{\deg A}(A \circ \nabla), \tag{22}$$

which is an element of $\Omega^*(M, \mathrm{End}(E))$ (with the action of each term interpreted accordingly). And $d(\mathrm{Tr}(A)) = \mathrm{Tr}([\nabla, A])$.

Proof. It can be checked locally that $d(\mathrm{Tr}(A)) = \mathrm{Tr}(dA)$. But locally $\nabla = d + B$ for some $B \in \Omega^1(M, \mathrm{End}(E))$, thus $[\nabla, A] = dA + [A, B]$, giving that $\mathrm{Tr}([\nabla, A]) = \mathrm{Tr}(dA)$. $\square$

Example 4.7.4. We have $[\nabla, \Omega] = [\nabla, \nabla^2] = \nabla \circ \nabla^2 - \nabla^2 \circ \nabla = 0$, so $d(\mathrm{Tr}(\Omega)) = \mathrm{Tr}([\nabla, \Omega]) = 0$, hence we see that $\mathrm{Tr}(\Omega)$ is a closed form in $\Omega^*(M)$.

In general, given a formal power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, we put

$$f(\Omega) := a_0\,\mathrm{Id}_E + a_1\Omega + a_2\Omega^2 + \cdots \in \Omega^*(M, \mathrm{End}(E)). \tag{23}$$

Theorem 4.6 (Chern-Weil)**.** *Given connection ∇ on a bundle $E \to M$, its curvature Ω and a power series $f(\Omega)$ of the curvature like above, then*

(i) $\mathrm{Tr}(f(\Omega)) \in \Omega^*(M)$ is a closed form.

(ii) The cohomology class $[\mathrm{Tr}(f(\Omega))] \in H^*_{\mathrm{dR}}(M)$ in independent of the connection.

Proof. For (i), we use the lemma and cook up a similar argument for $f(\Omega)$ as that for Ω in the above example.

Mainly we prove (ii). In fact, we shall show that if $\{\nabla^t\}$, $t \in [0,1]$ is any (smooth) family of connections, then

$$\mathrm{Tr}(f(\Omega^{(1)})) - \mathrm{Tr}(f(\Omega^{(0)})) = d\alpha_f, \quad \text{with } \alpha_f = \int_0^1 \mathrm{Tr}\left(\frac{\mathrm{d}\nabla^t}{\mathrm{d}t} f'(\Omega^{(t)})\right) dt, \tag{24}$$

and hence $[\mathrm{Tr}(f(\Omega^{(1)}))]$ and $[\mathrm{Tr}(f(\Omega^{(0)}))]$ will represent the same cohomology class. Here $\frac{\mathrm{d}\nabla^t}{\mathrm{d}t}$ is seen as an element in $\Omega^1(M, \mathrm{End}(E))$. Moreover, for any two connections $\nabla^{(1)}$ and $\nabla^{(0)}$, we can always "connect" them by a line segment $\nabla^{(t)} = t\nabla^{(1)} + (1-t)\nabla^{(0)}$, so formula (24) leads to our result.

To prove (24), we consider $\tilde{M} = M \times [0,1]$ with projection $\pi :\to M$ s.t. $(x,t) \mapsto x$, and put $\tilde{E} = \pi^*E$. Define connection $\tilde{\nabla}$ on $\tilde{E}$ by $\tilde{\nabla} = \nabla^{(t)} + \mathrm{d}t \wedge \frac{\partial}{\partial t}$. From (i) it follows that

$$\tilde{d}\left(\mathrm{Tr}(f(\tilde{\Omega}))\right) = 0, \tag{25}$$

where $\tilde{d} = d + \mathrm{d}t \wedge \frac{\partial}{\partial t}$ and $\tilde{\Omega} = \tilde{\nabla}^2 = \Omega^{(t)} + \mathrm{d}t \wedge \frac{\mathrm{d}\nabla^{(t)}}{\mathrm{d}t}$. After some calculation, we will be able to find

$$\mathrm{Tr}(f(\tilde{\Omega})) = \mathrm{Tr}(f(\Omega^{(t)})) + \mathrm{d}t \wedge \mathrm{Tr}\left(\frac{\mathrm{d}\nabla^{(t)}}{\mathrm{d}t} f'(\Omega^{(t)})\right). \tag{26}$$

Then expand (25) and look at the $\mathrm{d}t$ part we find

$$\frac{\partial}{\partial t}\mathrm{Tr}(f(\Omega^{(t)})) = d\left(\mathrm{Tr}\left(\frac{\mathrm{d}\nabla^{(t)}}{\mathrm{d}t} f'(\Omega^{(t)})\right)\right). \tag{27}$$

Thus the result follows from integration. $\square$

Remark. To compute formula (26) we write $f(\tilde{\Omega}) = a_0 \operatorname{Id}_{\tilde{E}} + a_1\tilde{\Omega} + a_2\tilde{\Omega} + \cdots$ where $\tilde{\Omega} = \Omega^{(t)} + \mathrm{d}t \wedge \frac{\mathrm{d}\nabla^{(t)}}{\mathrm{d}t}$, then $\left(\Omega^{(t)} + \mathrm{d}t \wedge \frac{\mathrm{d}\nabla^{(t)}}{\mathrm{d}t}\right)^k = (\Omega^{(t)})^k + \cdots$, etc., the only thing to note is that taking the trace commutes the wedge products, and hence allows terms to cancel.

Next we shall promote the set up to superconnections. Let $\sigma : E \to E$ be a $\mathbb{Z}_2$-grading, namely $\sigma^2 = \operatorname{Id}_E$, giving the decomposition $E = E^+ \oplus E^-$. For $A \in C^\infty(M, \operatorname{End}(E))$, we have the supertrace $\operatorname{Tr}_S(A) = \operatorname{Tr}(\sigma A)$. So naturally we can write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{28}$$

so $\operatorname{Tr}_S(A) = \operatorname{Tr}(A_{11}) - \operatorname{Tr}(A_{22})$. The definition extends to $\operatorname{Tr}_S : \Omega^*(M, \operatorname{End}(E)) \to \Omega^*(M)$ with $\operatorname{Tr}_S(\omega \otimes A) = \omega \operatorname{Tr}_S(A)$ upon noting that $\operatorname{End}(E)$ also has a $\mathbb{Z}_2$-grading $\operatorname{End}(E) = \operatorname{End}^+(E) \oplus \operatorname{End}^-(E)$, where $\operatorname{End}^+(E)$ is the block diagonal part and $\operatorname{End}^-(E)$ the block off-diagonal part.

Let $V \in C^\infty(M, \operatorname{End}^-(E))$, $\nabla = \nabla^{E^+} \oplus \nabla^{E^-}$, and put $\mathbb{A} = \nabla + V$, which is called the "superconnection" on E. Note that (we will show?) the curvature $\mathbb{A}^2$ is seen as an element of $\Omega^*(M, \operatorname{End}(E))$.

Example 4.7.5. We have constructed $E = \bigoplus_{i=0}^l E^i$ with $\sigma = (-1)^N$, N the number operator. Thus writing any $A = \sum_{i=0}^l A_i$ with $A_i \in \Omega^*(M, \operatorname{End}(E^i))$, we will have $\operatorname{Tr}_S(A) = \sum_{i=0}^l (-1)^i \operatorname{Tr}(A_i)$.

Analogously we have

Theorem 4.7 (Super Chern-Weil)**.** *Same setting as above, then*

(i) $\operatorname{Tr}_S(f(\mathbb{A}^2)) \in \Omega^*(M)$ *is a closed form.*

(ii) The cohomology class $[\operatorname{Tr}_S(f(\mathbb{A}^2))]$ *is independent of the superconnection.*

To prove it, we shall consider a similar identity to (27) for the family $\mathbb{A}_t = \nabla + tV$.

5 Chern-Weil Theorem on Complex Manifolds

Let $\{E_i\}$ be a finite family of complex vector bundles equipped with a Hermitian metric, and suppose we have a holomorphic chain complex

$$0 \xrightarrow{v} E_0 \xrightarrow{v} E_1 \xrightarrow{v} \ldots \xrightarrow{v} E_l \xrightarrow{v} 0$$

$v^2 = 0$ and v is a holomorphic line bundle homomorphism. Now let $E = \bigoplus_{i=0}^l E_i = E^+ \oplus E^-$ with $\mathbb{Z}_2$-grading $\sigma = (-1)^N$ where N is the number operator. This gives the usual decomposition $E = E^+ \oplus E^-$ with E^+ the direct sum over even i, E^- over odd i. This means that v, v^* and $V = v + v^*$ are maps

$$\begin{aligned} v :& E^\pm \to E^\mp, \\ v^* :& E^\mp \to E^\pm, \\ V :& E^\pm \to E^\mp, \end{aligned}$$

that is, each of v, v^*, V belong to the space $End^-(E)$ of block off-diagonal maps on E.

Let ∇^{E_i} be the Chern connection on E_i and define $\nabla = \bigoplus_{i=1}^l \nabla^{E_i}$. Akin to the real case, we now define family of super-connection $\mathbb{A}_a = \nabla + V^a$ for a complex-valued and $V_a = av + \overline{a}v^* \in End^-(E)$.

Theorem 5.1. *(1) For all $a \in \nabla$, $Tr_s[e^{-\mathbb{A}^2}]$ and $Tr_s[Ne^-\mathbb{A}^2]$ consist of sums of (p,p)-forms.*

(2) The first of the above expressions is both ∂- and $\overline{\partial}$-closed.

(3) The following identities hold:

$$\begin{aligned}
\frac{\partial}{\partial a} Tr_s[e^{-\mathbb{A}^2}] &= -\partial Tr_s[ve^{-\mathbb{A}^2}], \\
\frac{\partial}{\partial \overline{a}} Tr_s[e^{-\mathbb{A}^2}] &= -\overline{\partial} Tr_s[v^* e^{-\mathbb{A}^2}]; \\
\frac{\partial}{\partial a} Tr_s[ave^{-\mathbb{A}^2}] &= -\overline{\partial} Tr_s[Ne^{-\mathbb{A}^2}], \\
\frac{\partial}{\partial \overline{a}} Tr_s[\overline{a}v^* e^{-\mathbb{A}^2}] &= +\partial Tr_s[Ne^{-\mathbb{A}^2}].
\end{aligned}$$

In particular, the above imply

$$\begin{aligned}
\frac{\partial}{\partial a} Tr_s[e^{-\mathbb{A}^2}] &= \frac{1}{a}\partial\overline{\partial}\, Tr_s[Ne^{-\mathbb{A}^2}], \\
\frac{\partial}{\partial \overline{a}} Tr_s[e^{-\mathbb{A}^2}] &= -\frac{1}{\overline{a}}\overline{\partial}\partial\, Tr_s[Ne^{-\mathbb{A}^2}].
\end{aligned}$$

We remark here that each term of a complex differential form is has a degree specified by two parameters (p, q), with p the degree of its holomorphic part and q the degree of its anti-holomorphic part.

Proof. Observe that

$$(\nabla + V_a)^2 = \nabla^2 + |a|^2(vv^* + v^*v) + (a\nabla' v + \overline{a}\nabla'' v^*),$$

where $\nabla = \nabla' + \nabla''$ is the decomposition of ∇ into its holomorphic and anti-holomorphic parts. Note that ∇^2 is a $(1, 1)$-form, $|a|^2(vv^* + v^*v)$ is a $(0, 0)$-form, and $a\nabla' v + \overline{a}\nabla'' v^*$ is the sum of a $(1, 0)$- and a $(0, 1)$-form.

Therefore we can we can conclude (1). Also, using Chern-Weil and denoting $d = \partial + \overline{\partial}$, we have

$$dTr_s[e^{-\mathbb{A}^2}] = \partial Tr_s[e^{-\mathbb{A}^2}] + \overline{\partial} Tr_s[e^{-\mathbb{A}^2}] = 0,$$

which is a sum of forms of type $(p + 1, p)$ and $(p, p + 1)$. Hence, each term vanishes individually, so $Tr_s[e^{-\mathbb{A}^2}]$ is both ∂- and $\overline{\partial}$-closed, as claimed.

Now we move to the first two identities in (3). We apply the same procedure as in the real case, only this on $B \times \nabla$:

$$\begin{aligned}
\left(\partial + da\frac{\partial}{\partial a}\right) Tr_s\Big[\exp(\nabla + da\frac{\partial}{\partial a} + d\overline{a}\frac{\partial}{\partial \overline{a}} + V^a)^2\Big] &= 0, \\
\left(\overline{\partial} + d\overline{a}\frac{\partial}{\partial \overline{a}}\right) Tr_s\Big[\exp(\nabla + da\frac{\partial}{\partial a} + d\overline{a}\frac{\partial}{\partial \overline{a}} + V^a)^2\Big] &= 0.
\end{aligned}$$

But observe that $\nabla + da\frac{\partial}{\partial a} + d\overline{a}\frac{\partial}{\partial \overline{a}} = \mathbb{A}_a^2 + dav + d\overline{a}v^*$, which gives us

$$Tr_s\Big[\exp(\nabla + da\frac{\partial}{\partial a} + d\overline{a}\frac{\partial}{\partial \overline{a}} + V^a)^2\Big] = da \wedge d\overline{a} \wedge \varepsilon + Tr_s[e^{-\mathbb{A}^2}] - da\, Tr_s[ve^{-\mathbb{A}^2}] - d\overline{a}\, Tr_s[v^* e^{-\mathbb{A}^2}],$$

where ε is some differential form. Using one of the identities above, setting the da term to 0, we find

$$\frac{\partial}{\partial a} Tr_s[e^{-\mathbb{A}_a^2}] + \partial Tr_s[ve^{-\mathbb{A}_a^2}] = 0,$$

giving us the first identity in (3). The second follows in the same way after setting the $d\overline{a}$ term to 0.

Finally, we compute $dTr_s[Ne^{-\mathbb{A}_a^2}]$:

$$\begin{aligned}
dTr_s[Ne^{-\mathbb{A}_a^2}] &= ... \\
&= Tr_s[(-av + \overline{a}v^*)e^{-\mathbb{A}_a^2}],
\end{aligned}$$

giving us the second pair of identities in (3). □

6 Holomorphic determinant line bundles in infinite dimension

First we make some remarks about the real setting before returning to the complex setting of interest to us.

Let $\pi : M \to B$ be a submersion from a real total manifold M to a real base manifold B. Having a submersion π means that for each $y \in B$, the fibers $\pi^{-1}(y) = Z_y$ are smooth real manifolds diffeomorphic to each other. That is, each $x \in M$ with $\pi(x) = y$ has a neighborhood isomorphic to the product manifold $B \times \pi^{-1}(y)$ (local triviality). Setting $Z \cong Z_y$, this information amounts to a fiber bundle with the diagram

$$Z \hookrightarrow M \xrightarrow{\pi} B.$$

As we've previously done, we can define the vertical tangent bundle $T^V M \subset TM$ to be vector fields on M that are tangent to the fibers, or equivalently, the vector fields that descend via π to the trivial vector field on B. For simplicity we denote $TZ = T^V M$. We can further define a complementary subbundle $T^H M \subset TM$ such that $TM = T^H \oplus TZ$ with corresponding projections $p^H : TM \to T^H M$ and $p^Z : TM \to TZ$. We call $T^H M$ the horizontal tangent bundle.

We remark that $T^H M \cong \pi^* TB$, so any vector field in TB lifts to a vector field in $T^H M$, which we call the horizontal lift.

Now equip B, Z with smooth metrics g^B, g^Z. We can then define a smooth metric $g = \pi^* g^B + g^Z$ on the total manifold M, called the submersion metric.

Let $nabla^L$ denote the Levi-Civita connection on TM. We let $\nabla^Z = p^Z \nabla^L$ denote the projection of ∇^L on TZ and $R^Z = (\nabla^Z)^2$ denote the curvature on TZ, each of which is compatible with the metric g^Z by construction.

The machinery explained above largely passes to the complex setting, but with some subtleties which we mention here. For complex manifolds M, B, we now require the submersion $\pi : M \to B$ to be holomorphic. We also require the fibers $Z \cong Z_y$ to be compact, complex manifolds, and will later impose even stronger conditions. We now have a short-exact sequence of holomorphic vector bundles

$$0 \to T^{(1,0)} Z \to T^{(1,0)} M \to \pi^* T^{(1,0)} B \to 0.$$

The local triviality near a base point y of this vector bundle may not inherit a natural complex structure – the short exact sequence above may not split holomorphically.

We now introduce an important definition:

Definition 6.1. The triple $(\pi, g^Z, T^H M)$ is called a *Kähler fibration* if there exists a smooth $(1,1)$-form ω on M such that:

1. $d\omega = 0$;
2. for all $X \in TZ$ and $Y \in T^H M$, we have $\omega(X, Y) = 0$;
3. for all $X, Y \in TZ$, we have $\omega(X, Y) = \langle X, JY \rangle$, where $J^2 = -I$.

We remark that the form ω restricts to a Kähler form on the fibers of π, meaning that that the fibers are Kähler manifolds.

Example. Let (M, g) be a Kähler manifold with $g^Z = g\big|_{TZ}$, $T^H M = (TZ)^{\perp_g}$. By definition M is equipped with a Kähler form ω that is closed and satisfies (1) and (2) over TM. By restricting to TZ, we observe that the triple $(\pi, g^Z, T^H M)$ defines a Kähler fibration.

Example. Let (Z,ω) be a Kähler manifold and define $M = Z \times B \xrightarrow{\pi} B$ for any complex manifold B. Then (π, g^Z, ω) defines a Kähler fibration.

Given a Kähler fibration $(\pi, g^Z, T^H M)$, the connection ∇^Z turns out to be precisely the Chern connection on $T^{(1,0)}Z$. In particular, when M is a a Kähler manifold, the Levi-Civita connection and then Chern connection are the same.

Now let $\xi \to M$ be another holomorphic vector bundle.

Definition 6.2. For $0 \le p \le l$, we define $E^p = C^\infty(M, \bigwedge^p(T^{*(0,1)}Z)\otimes\xi)$, and we further define its restriction $E^p_y = C^\infty(Z_y, \bigwedge^p(T^{*(0,1)}Z_y) \otimes \xi\big|_{Z_y})$.

We can think of E^p an infinite-dimensional vector bundle.

For each $y \in B$, we have $\overline{\partial}^{Z_y} : E^p_y \to E^{p+1}_y$. We define $\overline{\partial}_y = \sqrt{2}\,\overline{\partial}^{Z_y}$. The factor $\sqrt{2}$ is a convention related to the identity $\Delta_d = 2\Delta_{\overline{\partial}}$, where $d = \partial + \overline{\partial}$.

Theorem 6.1. *There is a holomorphic line bundle $\lambda \to B$ such that for every $y \in B$, the fibers $\lambda_y \cong (DetH^{0,0}(Z_y, \xi\big|_{Z_y}))^{-1} \otimes (DetH^{0,1}(Z_y, \xi\big|_{Z_y}))^{-1} \otimes ....$*

Proof. Let $\langle,\rangle$ be a Hermitian structure on E^p. We can define an L^2 metric on global sections s, s' of E^p of the form

$$\int_{Z_y} \langle s, s'\rangle.$$

With respect to this metric, we have the adjoint $\overline{\partial}^*_y = \sqrt{2}\,(\overline{\partial}^{Z_y})^*$, and we can define a family of Dirac operators

$$D_y = \overline{\partial}_y + \overline{\partial}^*_y.$$

Next, for all $a \in \mathbb{R}$, we define $U^a = \{y \in B : a \notin SpecD^2_y\}$, which is the subset of B for which a is in the resolvent set of the Laplacian D^2_y. We further define $K^{a,p}$ to be the direct sum of eigenspaces of D^2_y with eigenvalue less than a, so that $K^{\infty,p} = L^2(E^p_y)$. For a fixed, we have the holomorphic line bundle

$$0 \to K^{a,0} \xrightarrow{\overline{\partial}} K^{a,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} K^{a,l} \to 0,$$

from which we obtain a determinant line bundle $\lambda_a \to U_a$. We can similarly define $K^{a,b,p} = \{y \in B : a \notin SpecD^2_y\}$ to be the direct sum of eigenspaces of D^2_y with eigenvalues in the interval (a,b), giving the holomorphic chain complex

$$0 \to K^{a,b,0} \xrightarrow{\overline{\partial}} K^{a,b,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} K^{a,b,l} \to 0.$$

This chain complex is in fact acyclic, so we obtain nowhere vanishing "holomorphic" sections $T(\overline{\partial}^{a,b}) \in \lambda^{a,b}$ on the determinant line bundle $\lambda^{a,b}$.

We remark that in general, if $\pi : M \to B$ is a holomorphic submersion where each fiber is a compact, complex manifold, the Hodge numbers $h^{p,q}(Z_y) = \dim H^{p,q}(Z_y)$ may not be constant in y. However, we have the following stability result: for any $y_0 \in B$ such that Z_{y_0} is Kähler, we have $h^{p,q}(Z_y) = h^{p,q}(Z_{y_0})$ for y near y_0. Therefore the Hodge numbers are constant in the case that each fiber is Kähler. For simplicity, we assume $\dim H^p(Z_y, \xi\big|_{Z_y})$ is constant, giving us a smooth vector bundle

$$\dim H^p(Z_y, \xi\big|_{Z_y}) \to y \in B.$$

Then we have a vector bundle isomorphism

$$\lambda \cong (DetH^0(Z_y, \xi\big|_{Z_y}))^{-1} \otimes (DetH^1(Z_y, \xi\big|_{Z_y}))^{-1} \otimes ...$$

$\square$

The Hodge theorem states that $H^p(Z_y, \xi\big|_{Z_y}) = ker D_y^2 \subset E_y^2$ inherits an L^2 metric, which in turn induces an L^2 metric on the determinant line bundle λ via the vector bundle isomorphism mentioned at the end of last class. We denote this L^2 metric by $|\cdot|$. We define the *Quillen metric* $\|\cdot\|$ to be

$$\|\cdot\| = |\cdot| \exp(-\frac{1}{2}\zeta_y'(0)),$$

where

$$\zeta_y = \frac{-1}{\Gamma(s)} \int_0^\infty u^{s-1} Tr_s(Nv \exp(-uD_y^2))\, du.$$

We now state theorem summarizing some important results about this metric.

Theorem 6.2 (Quillen, Bismut-Gillet-Soule)**.** *The Quillen metric defines a Hermitian metric on λ, the curvature of its Chern connection is given by*

$$2\pi i[\int_Z Td(-\frac{R^Z}{2\pi i}) \wedge Tr\left(\exp(-\frac{R^\xi}{2\pi i})\right)]_{(2)}.$$

Here $[\cdot]_{(2)}$ means taking the 2-form part. Since TZ is a vector bundle on M with connection ∇^Z defined above, while R^Z is the curvature with respect to ∇^Z. ξ is a holomorphic vector bundle on M with Chern connection ∇^ξ, R^ξ is the curvature with respect to ∇^ξ.

Remark 6.2.1. *This theorem has a lot of applications in many fields, including Arakalov Geometry and Mirror symmetry.*

Outline of the Proof. Notice that, if s is a local section of λ, then

$$R^{\|\cdot\|} = \partial\bar{\partial}\log\|s\|^2 = \partial\bar{\partial}\log|s|^2 - \partial\bar{\partial}\zeta'(0) = R^{\|\cdot\|} - \partial\bar{\partial}\zeta'(0),$$

i.e.

$$R^{\|\cdot\|} - R^{|\cdot|} = -\partial\bar{\partial}\zeta'(0),$$

which looks like double transgression formula.

Indeed, in finite dimensional case, let $\mathbb{A}_u = \nabla + \sqrt{u}V$, then we have

$$\frac{\partial}{\partial u} Tr_s[\exp(-\mathbb{A}_u^2)] = \frac{1}{u}\partial\bar{\partial} Tr_s[N\exp(-\mathbb{A}_u^2)],$$

which implies

$$Tr_s[\exp(-\mathbb{A}_u^2)]\big|_{u=\epsilon}^T = \partial\bar{\partial}\int_\epsilon^T \frac{1}{u} Tr_s[N\exp(-\mathbb{A}_u^2)]du.$$

Notice that $\mathbb{A}_u^2 = \nabla^2 + \sqrt{u}[\nabla, V] + uV^2$, $\lim_{u\to\infty} Tr_s[\exp(-\mathbb{A}_u^2)]_{(2)} = Tr_s[\exp(-\nabla^{0^2})]_{(2)}$, where ∇^0 is the projection of ∇ onto $\ker V^2 \cong H^*(E, V)$.

Moreover, $\lim_{u\to 0} Tr_s[\exp(-\mathbb{A}_u^2)] = Tr_s[\exp(-\nabla^2)]$.

Consequently, $Tr_s[\exp(-\nabla^2)] - Tr_s[\exp(-\nabla^{0^2})] = \partial\bar{\partial}\zeta'(0)$.

For infinite dimensional case, we consider $\mathbb{A}_u = \tilde{\nabla} + \sqrt{u}D$, where $\tilde{\nabla}$ is a connection on $E^p \to B$ defined as follows:

For any $Y \in C^\infty(B, TB)$, $s \in C^\infty(B, E^p) = C^\infty(M, \Lambda^P T^{*(0,1)}Z \otimes \xi)$,

$$\tilde{\nabla}_Y s := \nabla_{Y^H}^{Z\otimes\xi} s.$$

It turns out that $\tilde{\nabla}$ is the Chern connection on $E^p \to B$ with respect to L^2 metric.

In this case, we still have

$$\frac{\partial}{\partial u} Tr_s[\exp(-\mathbb{A}_u^2)]_{(2)} = \frac{1}{u}\partial\bar{\partial} Tr_s[N_v \exp(-\mathbb{A}_u^2)].$$

As before, $\lim_{u\to\infty} Tr_s[\exp(-\mathbb{A}_u^2)]_{(2)} = Tr_s[\exp(-\nabla^{0^2})]_{(2)} = R^{|\cdot|}$, where $\tilde{\nabla}^0$ is the projection of $\tilde{\nabla}$ on $\ker D^2 \cong H^{0,*}(Z, \xi_Z)$.

$\lim_{u\to 0} Tr_s[\exp(-\mathbb{A}_u^2)]_{(2)} = 2\pi i[\int_Z Td(-\frac{R^Z}{2\pi i}) \wedge Tr(\exp(-\frac{R^\xi}{2\pi i}))]_{(2)}$.

Hence, $R^{\|\cdot\|} = R^{|\cdot|} - \partial\bar{\partial}\zeta'(0) = 2\pi i[\int_Z Td(-\frac{R^Z}{2\pi i}) \wedge Tr(\exp(-\frac{R^\xi}{2\pi i}))]_{(2)}$. ∎

7 Mirror Symmetry at Higher Genus

7.1 Introduction

Mirror symmetry predicts some symmetry about Calabi-Yau manifold, Fano Manifold, Landau-ginzburg model etc.

Roughly speaking, we have

A Model	B Model
Symplectic Geometry	Complex Geometry
$F_g(Q) = \sum_{d\geq 0} N_{g,d}Q^d$	$g = 0$: Deformation of Complex Structure using Period integral $g \geq 1$: BCOV holomorphic anomaly formulas

Here $N_{g,d}$ is the Gromov-Witten invariants, which counts the number of holomorphic curves of genus g, degree d in Calabi-yau Manifold.

In particular, $g = 1$ holomorphic anomaly formula is almost equivalent to curvature formula for Quillen metric. Moreover, holomorphic anomaly formula of higher genus is related to that of $g = 1$.

7.2 Calabi-Yau manifolds and their moduli

First of all, Kahler manifold is a complex manifold with an Hermitian metric $g = g_{i\bar{j}}dz^i \otimes d\bar{z}^j$, whose Kahler form $w = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is closed.

A Key feature of Kahler manifold is

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}},$$

where

$$\begin{aligned}
\Delta_d &= dd^* + d^*d \text{ (Hodge Laplacian);}\\
\Delta_\partial &= \partial\partial^* + \partial^*\partial;\\
\Delta_{\bar{\partial}} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \text{ (Dolbeault Laplacian).}
\end{aligned}$$

This implies the so called Hodge decomposition for compact Kahler manifold:

$$H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M).$$

Remark 7.0.1. *Since Δ_d is real, we have $H^{p,q} = \overline{H^{q,p}}$, which implies $h^{p,q} = h^{q,p}$. Here $h^{p,q} = dimH^{p,q}$ is so called Hodge numbers.*

By Poincare duality, we have $H^{p,q} \cong \overline{H^{n-p,n-q}}$, where $n = dim_{\mathbb{C}}M$.

Example 7.0.1.

1. $\mathbb{C}^n$ with canonical metric $g_0 = \sum_{i=0}^n dz_i \otimes d\bar{z}^j$ is Kahler. Moreover, let $\Gamma \subset \mathbb{C}^n$ be a lattice of rank $2n$, then $\mathbb{C}^n/\Gamma$ is a compact Kahler manifold.

2. $\mathbb{C}P^n = \{[z_0, z_1, ...z_n]\}$ with Fubini-Study metric g_{FS} is also Kahler, where locally,

$$w_{FS} = \sqrt{-1}\partial\bar{\partial}\log(\sum_{i=0}^{b} |z_i|^2).$$

3. Any Complex submanifold of a Kahler manifold is also Kahler.

Definition 7.1. A Calabi-Yau manifold is a Kähler manifold M whose canonical line bundle $Det(T^{*(1,0)}M)$ is holomorphically trivial.

Example 7.1.1. The previous Kähler manifolds $\mathbb{C}^n$ and $\mathbb{C}^n/\Gamma$ are also examples of Calabi-Yau manifolds.

Example 7.1.2. Consider $\mathbb{CP}^n$ with homogeneous coordinates $[z_0, z_1, \ldots, z_n]$, and let $F(g)$ be a homogeneous polynomial of degree d and whose only cut point is 0. Then the locus $F(z_0, \ldots, z_n) = 0$ defines a Kähler manifold. In fact we have the following from the adjunction formula:

$$F = 0 \text{ is Calabi-Yau} \iff d = n + 1.$$

This gives us two more explicit examples of Calabi-Yau manifolds:

$$z_0^4 + \cdots + z_3^4 = 0 \in \mathbb{CP}^3;$$
$$z_0^5 + \cdots + z_4^5 = 0 \in \mathbb{CP}^4.$$

More generally, for a sufficiently small t, we have

$$z_0^4 + \cdots + z_3^4 + tz_0 \ldots z_3 = 0 \in \mathbb{CP}^3,$$
$$z_0^5 + \cdots + z_4^5 + tz_0 \ldots z_4 = 0 \in \mathbb{CP}^4$$

are Calabi-Yau.

We can construct a moduli space of Calabi-Yau structures on a manifold by identifying the Calabi-Yau structures which are biholomorphic. We say that M_1 and M_2 are biholomorphic if there exists holomorphic $F : M_1 \to M_2$ with a holomorphic inverse.

Example 7.1.3. **Elliptic curves.** Let $\Gamma \subset \mathbb{C}$ where $\Gamma = \langle \tau_1, \tau_2 \rangle$ and τ_1, τ_2 are linearly independent (here $\mathbb{C}$ is thought of as a 2-dimensional vector space of $\mathbb{R}$). We can assume that $\tau_1 = 1$ and $\tau_2 = \tau$ where τ lies in the upper-half plane $\mathbb{H}$. Unsurprisingly, we find that the elliptic curves corresponding to $\langle 1, \tau \rangle$ and $\langle 1, \tau' \rangle$ are biholomorphic if and only if there is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Exercise. Any holomorphic map $F : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ is induced by $\hat{\Gamma} : \mathbb{C} \to \mathbb{C}$ which is linear.

Thus, $\mathbb{H}/SL_2(\mathbb{Z})$ is the moduli space of 1-dimensional complex tori. We define $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$.

Example 7.1.4. Consider $z_0 + z_1^2 + z_2^2 = 0$ in $\mathbb{CP}^2$. This is a 1-dimensional Calabi-Yau, so so it must be associated with some $\tau \in \mathbb{H}/SL_2(\mathbb{Z})$. But which one? τ can be computed via the so-called "period integral." E_τ has a nowhere-vanishing holomorphic 1-form dz, called the Calabi-Yau form (it is not exact on E_τ).

$H_1(E_\tau)$ has a basis given by $a : Z(t) = t$, $b : Z(t) = t\tau$ for $t \in [0, 1]$. The pairing between dz and a, b gives period integrals

$$\pi_a = \oint_a dz = 1$$
$$\pi_a = \oint_b dz = \tau.$$

Now if we set $\mathfrak{X} = \{(\tau, [z]) : \tau \in \mathbb{H}, [z] \in \mathbb{C}/\langle 1, \tau \rangle\}$ and define the moduli space $\mathcal{M} = \mathbb{H}/SL_2(\mathbb{Z})$, the projection map

$$\pi : \mathfrak{X} \to \mathcal{M}, \qquad (\tau, [z])$$

is a holomorphic submersion fibers, where the fibers are denoted $E_\tau = \mathfrak{X}_\tau = \pi^{-1}(\tau)$.

We have seen some particularly simple examples of Calabi-Yau manifolds. However, things are typically much more difficult:

- The moduli spaces can be bad.
- Higher-dimensional Calabi-Yaus do not typically admit an explicit description.

To inform our understanding of other Calabi-Yaus, we explore deformation theory. Specifically, we look at deformations of complex structures using the Kodaira-Spencer maps. Some more definitions are necessary here.

Definition 7.2. An almost complex structure (acs) is a fiberwise map $J : TM \to TM$ with $J^2 = -id$. We say that it is integrable if either of the following conditions hold

$$\begin{cases} [T^{(1,0)}M, T^{(1,0)}M] \subset T^{(1,0)}M \text{ i.e. } T^{(1,0)}M \text{ is integrable} \\ N_J = 0 \text{ i.e., the so-called Nijenhaus tensor vanishes} \end{cases}$$

It is a theorem due to Newlander-Niremberg that the above two conditions are equivalent.

Definition 7.3. A complex structure on M^{2n} is an almost-complex structure that is integrable.

Any complex manifold has a canonical almost-complex structure which is integrable. The converse is proved via the Newlander-Nirenberg theorem which states that J is integrable if and only if the Nijenhuis tensor N_J vanishes.

Let J be a complex structure. A deformation of complex structures is a continuous family of almost complex structures $J(t)$ for $t \in (-\varepsilon, \varepsilon)$ such that $J(0) = J$, $N_{J(t)} = 0$. We write the infintesimal deformation by

$$\eta = \frac{dJ(t)}{dt}\Big|_{t=0} \in End(TM) = T^*M \otimes_{\mathbb{R}} TM$$

We note that $N_{J(t)} = 0$ implies that $\bar{\partial}_{T^{(1,0)}M}\eta = 0$. Hence, $\eta \in ker(\bar{\partial}_{T^{(1,0)}M})$, so this kernel will track all the deformations of complex structures.

We now identify biholomorphic complex structures at the infintesimal level to get the formal tangent space of complex structures. Let V be a vector field on M, and let $F(t) : M \to M$ be the flow (1 parameter family of diffeomorphisms) generated by V. Via pullback, we may obtain the following family of complex structures:

$$J(t) = (DF(t))^{-1} \circ J \circ DF(t)$$

By construction, $(M, J(t))$ and (M, J) have a biholomorphism given by $F(t)$. Then we have that

$$\eta = \frac{dJ(t)}{dt}\Big|_{t=0} = \bar{\partial}v_n$$

where $v_n \in C^\infty(M, T^{(1,0)}M)$ is given by v. Thus, we see that $\eta \in in(\bar{\partial}_{T^{(1,0)}M})$ will be deformations of complex structures in the same biholomorphic class. Thus, the formal tangent space of complex structures at J will be contained in $ker(\bar{\partial}_{T^{(1,0)}M})/im(\bar{\partial}_{T^{(1,0)}M}) = H^{(0,1)}(M, T^{(1,0)}M)$.

In fact, the formal tangent space of the moduli space of complex structures equals $H^{0,1}(M, T^{(1,0)}M$ in certain nice cases. However, in full generality there is some subtlety. The most important of these is the issue of obstructedness: a cohomology class $[\eta] \in H^{0,1}(M, T^{(1,0)}M)$ may not actually arise from a family of complex structures. Fortunately, this problem disappears because of the Calabi-Yau condition because its canonical line bundle vanishes. This is stated in the following theorem:

Theorem 7.1. *(Tian-Todorov). For a complex Calabi-Yau with $H^{0,0}(M, T^{(1,0)}M) = 0$ (this condition implies biholomorphisms are discrete), a deformation of its complex structure is unobstructed. In particular, the universal moduli space of complex structures is a smooth complex manifold with tangent space $H^{0,1}(M, T^{(1,0)}M)$.*

Remark 7.1.1. *A Calabi-Yau M^n with $\bigwedge^n T^{*(1,0)}M$ trivial implies $T^{(1,0)}M \cong \bigwedge^{n-1} T^{*(0,1)}M$, which implies $H^{0,1}(M, T^{*(0,1)}M) = H_{n-1,1}M$.*

Example 7.3.1. If M is a 1-dimensional Calabi-Yau, $\dim H^{1-1,1}M = H^{0,1}M = h^{0,1} = 1$, which agrees with our previous discussion about elliptic curves.

7.3 Weil-Petersson geometry of moduli space

For simplicity, M is a compact Calabi-Yau 3-fold. Thus there exist a nowhere-vanishing holomorphic (3,0)-form Ω, referred as the Calabi-Yau 3-form. It is unique up to constant multiplication.
Denote $\mathcal{M}_M$ as moduli space of complex structures on M. By Kodaira-Spencer, this is a complex manifold. Turns out that $\mathcal{M}_M$ is Kahler, with a canonical choice of metric, called the Weil-Petersson metric.
Indeed, $T\mathcal{M}_M \cong H^{0,1}(M, T^{(1,0)}M)$.
For all $u \in H^{0,1}(M, T^{(1,0)}M)$, we have $i_u\Omega \in \Omega^{2,1}(M)$, where i denotes contraction. Then the Weil-Petersson metric is $\forall u, v \in H^{0,1}(M, T^{(1,0)}M)$,

$$(u,v)_{WP} = \frac{\int_M (i_u\Omega) \wedge \overline{i_v\Omega}}{\int_M \Omega \wedge \bar{\Omega}}.$$

Exercise. For $\tau \in \mathbb{H}/SL(2,\mathbb{Z})$, $E_\tau = \mathbb{C}/\langle 1, \tau\rangle$, compute $\frac{\partial}{\partial \tau}$ as an element of $H^{0,1}(E_\tau, T^{(1,0)}E_\tau)$, and hence $\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau}\right)_{WP}$. (the answer should be the hyperbolic metric.)

Turns out that there is another way to look at the Weil-Petersson metric.
Let $\pi : \mathcal{X} \to \mathcal{M}_M$ be a universal deformation. For each $\tau \in \mathcal{M}_M$, $\mathcal{X}_\tau = \pi^{-1}(\tau)$ is another Calabi-Yau manifold. Therefore $H^{3,0}(\mathcal{X}_\tau) \cong \mathbb{C}$, trivialized by a choice of Calabi-Yau form Ω_τ. This gives rise to a holomorphic line bundle $L \to \mathcal{M}_M$ with $L_\tau = H^{3,0}(\mathcal{X}_\tau)$, $\tau \in \mathcal{M}_M$. Physicists usually call this the vacuum line bundle.
This line bundle has a natural Hermitian metric

$$||\Omega_\tau||^2 = (\Omega_\tau, \Omega_\tau) = \sqrt{-1}\int_{\mathcal{X}_\tau} \Omega_\tau \wedge \overline{\Omega_\tau}.$$

Hence the curvature of its Chern connection is given by

$$\partial\bar{\partial}\log||\Omega||^2,$$

where Ω is a local holomorphic section of L.
Now we have the miracle:

Theorem 7.2. $\omega_{WP} = \partial\bar{\partial}\log||\Omega||^2$.

Corollary 7.2.1. *Weil-Petersson metric is Kahler.*

Proof. (Idea of proof) Let $\tau = (\tau^a)$ be a local coordinate chart of $\mathcal{M}_M$. We have that $\partial_a\Omega$ is equal to a (3,0)-piece and a (2,1)-piece, which must equal to $K_a\Omega + \chi_a$. Here $\chi_a = i_{\partial/\partial\tau^a}\Omega$.
Therefore,

$$\begin{aligned} &\partial_a\bar{\partial}_b \log||\Omega||^2 \\ &= \partial_a\left[\frac{-1}{\int\Omega\wedge\bar{\Omega}}\int\Omega\wedge\overline{\partial_b}\bar{\Omega}\right] \\ &= \left[-\frac{1}{\left(\int\Omega\wedge\bar{\Omega}\right)^2}\int\partial_a\Omega\wedge\bar{\Omega}\int\Omega\wedge\overline{\partial_b}\bar{\Omega}\right] + \left[\frac{1}{\int\Omega\wedge\bar{\Omega}}\int\partial_a\Omega\wedge\overline{\partial_b}\bar{\Omega}\right]. \end{aligned}$$

We claim that the above equals to

$$\frac{\int\chi_a\wedge\overline{\chi_b}}{\int\Omega\wedge\bar{\Omega}}.$$

$\square$

7.4 BCOV Torsion and Holomorphic Anomaly Equation

Let M be a compact Kahler manifold, with $\dim_{\mathbb{C}} M = n$. $\xi \to M$ is a holomorphic vector bundle. We have the determinant line bundle

$$\lambda(\xi) = \otimes_{q=0}^{n} (\mathrm{Det}\, H^{(0,q)}(M, \xi))^{(-1)^q}.$$

Note: this is the dual of the previous one.
It comes with a natural hermitian metric, the Quillen metric. Now, if $\pi : M \to \mathcal{X} \to \mathcal{M}$ is a Kahler fibration with typical fibre M.
$\xi \to \mathcal{X}$ being holomorphic implies $\lambda(\xi) \to \mathcal{M}$ being holomorphic line bundle, and the curvature of the Quillen metric is

$$-2\pi i \left[\int_M Td\left(\frac{-R^M}{2\pi i}\right) \wedge ch\left(\frac{-R^\xi}{2\pi i}\right) \right]^{[1,1]},$$

where R^M is curvature of vertical holomorphic tangent bundle.
$T^{(1,0)}M \subset T^{(1,0)}\mathcal{X}$. (For simplicity, we drop "(1,0)" notation later on), and

$$ch\left(-\frac{R^\xi}{2\pi i}\right) = Tr\left(\exp - \frac{R^\xi}{2\pi i}\right).$$

The standard notations are: For any complex vector bundle $\xi \to M$ of rank k,

$$c_1(\xi) = Tr\left(-\frac{R^\xi}{2\pi i}\right), \; c_k(\xi) = \det\left(-\frac{R^\xi}{2\pi i}\right).$$

More generally,

$$\det\left(I + t\frac{-R^\xi}{2\pi i}\right) = 1 + \sum_{i=1}^{k} c_i(\xi) t^i,$$

where $c_i(\xi)$ is the i-th Chern class.
For a line bundle, $c_1 = \frac{-R}{2\pi i}$.
Hence the curvature formula for Quillen metric is

$$c_1(\lambda(\xi)) = \left[\int_M Td\left(\frac{-R^{TM}}{2\pi i}\right) ch\left(\frac{-R^\xi}{2\pi i}\right) \right]^{(1,1)}.$$

Here $R^M = R^{TM}$.
BCOV made the following choice for ξ:

$$\xi = \oplus_{p=1}^{n} (-1)^p p \wedge^p T^*M.$$

This implies

$$\lambda = \otimes_{p,q=0}^{n} (Det H^{p,q}(M))^{(-1)^{p+q}p} \to \mathcal{M}_M,$$

which is called the BCOV line bundle.

Theorem 7.3. *BCOV With the induced Quillen metric,*

$$c_1(\lambda) = -\frac{1}{12}\left[\int_M c_1(TM) \cdot c_n(TM) \right]^{(1,1)}$$

Proof. We first note that by Bismut-Cheeger-Soule, we may express the right hand side as

$$c_1(\lambda) = \left[\int_M Td(TM) \cdot \sum_{p=0}^{n} (-1)^p p ch(\Lambda^p T^*M) \right]^{(1,1)} \tag{29}$$

Now, we will show that for any rank n bundle $\xi \to \mathfrak{X}$, we have the following equality:

$$Td(\xi) \cdot \sum_{p=0}^{n} (-1)^p ch(\Lambda^p \xi^*) = c_n(\xi) \tag{30}$$

To see that this is true, let us first diagonalize the hermitian matrix $-R^\xi/(2\pi\sqrt{-1})$ into the diagonal matrix with entries $\gamma_1, \cdots, \gamma_n$. These are the so-called Chern roots. We then have the following equalities via direct evaluation

$$\begin{cases} Td(\xi) = \sum_{i=1}^n \frac{\gamma_i}{1-e^{\gamma_i}} \\ ch(\xi) = \sum_{i=1}^n e^{\gamma_i} \\ c_n(\xi) = \prod_{i=1}^n \gamma_i \\ ch(\Lambda^p \xi^*) = \sum_{i_1,\cdots,i_p}^n \exp - \sum_{j=1}^p \gamma_{i_j} \end{cases} \tag{31}$$

Thus, we have the following equality

$$\sum_{p=0}^n (-1)^p ch(\Lambda^p \xi^*) = \sum_{p=1}^n \sum_{i_1,\cdots,i_p}^n \exp - \sum_{j=1}^p \gamma_{i_j} = \prod_{i=1}^n (1 - e^{-\gamma_i})$$

And plugging this and the first equality in the previous equation into (30) gives the desired equality. We may now repeat the same process used above to obtain the equation

$$\sum_{p=0}^n (-1)^p x^p ch(\Lambda^p \xi^*) = \prod_{i=1}^n (1 - xe^{-\gamma_i})$$

Differentiation this at $x = 1$, we obtain

$$\sum_{p=0}^n (-1)^p p ch(\Lambda^p \xi^*) = \frac{d}{dx} \prod_{i=1}^n (1 - xe^{-\gamma_i})\Big|_{x=1} = n \prod_{i=1}^n (1 - e^{-\gamma_i}) - \sum_{i=1}^n \prod_{j \neq i} (1 - e^{-\gamma_j})$$

Thus,

$$Td(\xi) \cdot \sum_{p=0}^n p(-1)^p ch(\Lambda^p \xi^*) = nc_n(\xi) - \sum_{i=1}^n \frac{\gamma_i}{1 - e^{\gamma_i}} \prod_{j \neq i} \gamma_j$$

Now, noting the formal series expansion

$$\frac{1}{1 - e^x} = 1 + \frac{x}{2} + \sum_{i=1}^\infty \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i}$$

where B_i is the ith Bernoulli number, the previous equation becomes

Note that $B_2 = 1/6$, which is where the $1/12$ comes from. Now, in equation (29), we only care about the (1,1) part, and since n degrees get integrated out, we only want the degree $(n + 2)$ part of $Td(\xi) \cdot \sum_{p=0}^n p(-1)^p ch(\Lambda^p \xi^*)$. This is precisely $-\frac{1}{12} c_n(\xi) c_1(\xi)$; plugging this equality into Bismut-Cheeger-Soule with $\xi = TM$ gives the desired result.

□

Now, from the Calabi Yau condition, the canonical bundle $\Lambda^n T^* M$ is holomorphically trivial. Thus,

$$c_1(M) = c_1(TM) = -c_1(T^*M) = -c_1(\Lambda^n T^* M) = 0$$

We also note Yau's celebrated theorem,

Theorem 7.4. *Let M be a Calabi Yau manifold. Then for each Kähler class, there exists a unique Ricci-flat metric g with Kähler form in the Kähler class. This is known as the calabi-Yau metric*

Now, suppose that our universal deformation $\mathfrak{X} \to \mathcal{M}$ has fiberwise Calabi Yau metric. Then,

Theorem 7.5. $c_1(\lambda_{BCOV}) = \frac{\chi(M)}{12} \omega_{WP}$

Proof. By the previous theorem,

$$c_1(\lambda_{BCOV}) = -\frac{1}{12}\left[\int_M c_1(TM)\cdot c_n(TM)\right]^{(1,1)}$$

Now, note that

$$c_1(TM) = -c_1(\Lambda^n T^*M) = -c_1(\pi^* L) = -\pi^* c_1(L) = -\pi^*\omega_{WP}$$

Here, L is a line bundle determined by the fiberwise Calabi Yau metric. The first equality follows by triviality of the canonical bundle, and the third by functoriality of the Chern classes, and the last by construction. Now noting that $-\pi^*\omega_{WP}$ has no vertical component, we have that

$$c_1(\lambda_{BCOV}) = -\frac{1}{12}\left[\int_{\mathcal{M}} -\pi^*\omega_{WP}\cdot c_n(TM)\right]^{(1,1)} = \frac{-\omega_{WP}}{12}\int_M c_n(TM) = \frac{-\omega_{WP}}{12}\chi(M)$$

as desired □

Now let us reinterpret what we have done so far. Recall that

$$\lambda_{BCOV} = \bigotimes_{p,q} Det(H^{p,q}(M))^{(-1)^{p+q}p}$$

And that each $H^{p,q}(M) \cong ker\Delta_{\bar{\partial}}\big|_{\Omega^{p,q}}$ as a C^∞ vector bundle over $\mathfrak{X}$. Here, $ker_{\bar{\partial}}$ is the Dolbeault Laplacian. Over this, there is a natural L^2 metric from the determinant of the Laplacian.

Definition 7.4. The kth Hodge form is

$$\omega_{H^k} = \sum_{p=0}^{n} p\cdot c_1(H^{p,k-p}) = \sum_{p=0}^{n} c_1(\mathcal{F}^p H^k)$$

Here $\mathcal{F}$ is the Hodge filtration.

And now we have the relation between the Quillen metric on Λ_{BCOV} and the torsion

$$\|\cdot\|_Q^2 = \|\cdot\|_{L^2}^2 T_{BCOV}$$

Recall that

$$T_{BCOV} = \prod_{p,q=1}^{n} Det(\Delta_{\bar{\partial}}^{p,q})^{(-1)^{p+q}pq}$$

Then, what we have done is that

Theorem 7.6. *(Genus 1 holomorphic anomaly formula)*

$$\partial\bar{\partial} log T_{BCOV} = \sum_{k=0}^{2n}(-1)^k\omega_{H^k} - \frac{\chi(M)}{12}\omega_{WP}$$

Proof. From above, we have that

$$\frac{\chi(M)}{12}\omega_{WP} = c_1(\lambda_{BCOV})$$

But by construction of the metric on λ_{BCOV}, it now follows that

$$\begin{aligned}
c_1(\lambda_{BCOV}) &= \sum_{p=0}^{n}(-1)^{p+q}pc_1(H^{p,q}) - \partial\bar{\partial}logT_{BCOV} \\
&= \sum_{k=0}^{2n}(-1)^k\sum_{p=0}^{k}pc_1(H^{p,k-p}) - \partial\bar{\partial}logT_{BCOV} \\
&= \sum_{k=0}^{2n}(-1)^k\omega_{H^k} - \partial\bar{\partial}logT_{BCOV}
\end{aligned}$$

and we are done. □

This theorem leads into the BCOV Conjecture

Conjecture (BCOV) Let $(X, X^\vee)$ be a Mirror pair of Calabi Yau 3-folds. Let $\mathcal{M}$ be the moduli space of complex structures over X. Finally, let $L \to \mathcal{M}$ be the vacuum line bundle. Then

1. There exists a C^∞ section $\mathcal{F}_g \in C^\infty(\mathcal{M}, L^{2-2g})$ called the genus g topological string amplitude
2. $\mathcal{F}_g$ satisfies the BCOV holomorphic anomaly formula:

$$\partial\bar{\partial}\mathcal{F}_g = \sum_{k=0}^{2n}(-1)^k\omega_{H^k} - \frac{\chi(M)}{12}\omega_{WP}$$

   and $\bar{\partial}\mathcal{F}_g$ satisfies a certain recursive formula in terms of the $g-1$ and $g-2$ amplitudes
3. There exists a procedure of passing to the holomnorphic limit to obtain a holomorphic section $\mathcal{F}_g \in H^0(\mathcal{M}, L^{2-2g}$
4. The gromov witten potential $\mathcal{F}_g(Q)$ of $X^\vee$ is obtained from $\mathcal{F}_g$ via the Mirror map.

Remark. 1. This is the Topological string formulation of Mirror Symmetry for genus $g \geq 1$.

2. There is also a homological mirror symmetry formuklation of Kontsevich
3. Furthermore, there is also the SYZ construction of Mirror pairs.