241C22 Scalar Curvature, Minimal Surfaces, and Spin

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March 2022

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1 Introduction

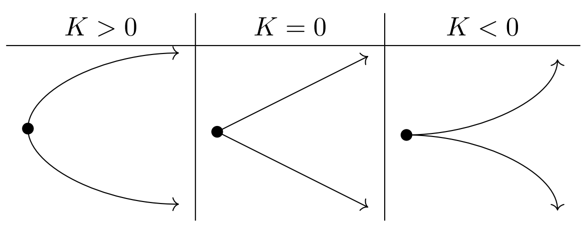
There is an interaction between geometry and topology. In other words, local geometric data is related to global topological information. One of the local geometric data is curvature. Typically, there are three curvatures: Sectional curvature, Ricci curvature, and Scalar curvature.

To calculate sectional curvature, we need to take two directions, then we get a number. Basically, we take a two dimensional section of a space and the scalar curvature is the Gaussian curvature of the section. On the other hand, for Ricci curvature, we fix one direction and then sum the sectional curvatures for each Riemannian perpendicular direction. In this sense, sectional curvature is much stronger than Ricci curvature. In the same way, Scalar curvature is obtained by the sum of Ricci curvatures for each Riemannian perpendicular direction. This implies that scalar curvature is not dependent on the direction of a vector, but dependent on only a point. So, Scalar curvature is weaker condition than Ricci curvature.

For the global topological information side, we usually look at the compactness of a space, Fundamental group π_1, homology, cohomology H_*. The below is the diagram for the interaction between geometry and topology.

$$\begin{array}{rcl} \text{Geometry} & \longleftrightarrow & \text{Topology} \\ \text{Local geometric data} & \longleftrightarrow & \text{global topological information} \\ \text{curvatures} & & \text{Compactness}, \pi_1, H_* \\ (\text{Sectional} > \text{Ricci} > \text{Scalar}) & & \end{array}$$

Jacobi equation describes the variation of geodesics. Sectional curvature controls how geodesics from a point spread:



"Philosophical Principle" Positively curved spaces tend to close up while non-positively curved spaces tend to open up. Indeed, in 240B, we have Cartan-Hadamard Theorem

Theorem 1.1. *Let (M^n, g) be a complete simply connected manifold with $K \leq 0$. Then M^n is diffeomorphic to $\mathbb{R}^n$.*

Remark. Let (M, g) be a complete manifold with $K \leq 0$: Apply to $(\tilde{M}, \tilde{g})$.

In contrast, we have Bonnet - Myers Theorem:

Theorem 1.2. *Let (M^n, g) be a complete manifold with $\mathrm{Ric} \geq (n-1)\kappa > 0$. Then M is compact and $\pi_1(M)$ is finite. In particular, if M is compact and $\mathrm{Ric} > 0$, then $\pi_1(M)$ is finite.*

Remark.

e.g. $\mathbb{S}^2 \times \mathbb{S}^1$ cannot have a metric with $\mathrm{Ric} > 0$.

How do we go about proving it? By Hopf-Rinow to show M is compact we just need to show it's bounded.

Claim: $\mathrm{Diam}(M) = \sup_{p,q \in M} d(p,q) \leq \frac{\pi}{\sqrt{\kappa}}$

More quantitative (and geometric) statement

How do we do that? Again by Hopf Rinow, any two points $p, q \in M$ can be connected by a **minimal** geodesic.

Show that if a geodesic is too long ($l > \frac{\pi}{\sqrt{\kappa}}$), then it CANNOT be minimal.

Recall. γ is a geodesic iff γ is a critical point of the length functional

$$L(\alpha) = \int_a^b |\alpha'(t)| dt, \quad (\alpha : [a,b] \to M \text{ is a } C^1 \text{ curve.})$$

among variations fixing the endpoints.

Recall. In calculus, we know that

$$\begin{array}{ccccc} \text{minimum} & > & \text{local minimum} & > & L'' \geq 0 \\ \text{(hard) - global} & & & & \text{(much easier)} \end{array}$$

Fact (2nd variation formula). Let $\gamma : [a,b] \to M$ is a geodesic. Then a smooth map $\alpha : [a,b] \times (-\epsilon, \epsilon) \to M$ with $\alpha(t,0) = \gamma(t)$, $\alpha(a,s) = \gamma(a)$, and $\alpha(b,s) = \gamma(b)$ is a variation of the given geodesic. Let $V(t) = \frac{\partial \alpha}{\partial s}(t,0)$ be a variational vector field. Assume that $V(t) \perp \gamma'(t)$. Then

$$L'' = \frac{d^2}{ds^2} L(\alpha(\cdot, s))\Big|_{s=0} = \int_a^b (|\nabla_{\gamma'} V|^2 - \langle R(V, \gamma')\gamma', V \rangle) dt$$

Remark. $\alpha \longleftrightarrow V$

Goal: If γ is a geodesic, unit speed, $l = L(\gamma) > \frac{\pi}{\sqrt{\kappa}}$, then there is V as above such that $L'' < 0$.

Proof of Bonnet-Myers. For the moment, assume in fact, $K \geq \kappa > 0$. How to pick V so that $L'' < 0$?

(figure)

Let

$$V(t) = \sin\left(\frac{\pi}{l} t\right) E(t).$$

Then

$$\begin{aligned} L'' &= \int_0^l \left(\left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l} t\right) - K(E, \gamma') \sin^2\left(\frac{\pi}{l} t\right) \right) dt \\ &\leq \left(\frac{\pi}{l}\right)^2 \frac{l}{2} - \kappa \frac{l}{2} \\ &= \frac{l}{2} \left[(\left(\frac{\pi}{l}\right)^2 - \kappa \right] < 0 \end{aligned}$$

because $l > \frac{\pi}{\sqrt{\kappa}}$.

For the general case, let $e_1, \cdots, e_{n-1}, \gamma'(0)$ be orthonormal basis of $T_{\gamma(0)M}$ and $E_1(t), \cdots, E_{n-1}(t)$ are parallel along γ. Put

$$V_i(t) = \sin\left(\frac{\pi}{l} t\right) E_i(t), \quad i = 1, \cdots, n-1$$

Denote $L_i'' =$ 2nd variation along V_i. Then

$$\sum_{i=1}^{n-1} L_i'' = \int_0^l \left((n-1) \left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l} t\right) - \sum_{i=1}^{n-1} K(E_i, \gamma') \sin^2\left(\frac{\pi}{l} t\right) \right) < 0$$

if $l > \frac{\pi}{\sqrt{\kappa}}$. Then, there is i such that $L_i'' < 0$. Thus, γ cannot be (even locally) minimal. $\square$

Remark. For $\pi_1(M)$ finite, apply to $(\tilde{M}, \tilde{g})$.

2 Variations of Area Functionals

Let M be a Riemannian manifold of dimension m and N be an n-dimensional submanifold with $n < m$. Consider an one parameter family of deformation of N given by $N_t = \phi(N, t)$ for $t \in (-\epsilon, \epsilon)$ with $N_0 = N$.

Let $\{x_1, \cdots, x_n\}$ be a coordinate system around a point $p \in N$. Then $\{x_1, \cdots, x_n, t\}$ is a coordinate system of $N \times (-\epsilon, \epsilon)$ near $(p, 0)$. Denote

$$e_i := d\phi\left(\frac{\partial}{\partial x_i}\right), \quad T = d\phi\left(\frac{\partial}{\partial t}\right).$$

2.1 First variational formula for area

Notation. dA_t:= the area element of N_t.

Proposition 2.1. *In local coordinates $\{x_1, \cdots, x_n\}$ of $p \in N$,*

$$dA_t = \sqrt{g(x,t)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

where $g(x,t) = \det(g_{ij})(x,t)$.

Proposition 2.2. *For t sufficiently close to 0,*

$$dA_t = J(x,t)dA_0.$$

In particular, if $\{x_1, \cdots, x_n\}$ is a normal coordinate system, then the function $J(x,t)$ is given by

$$J(x,t) = \frac{\sqrt{g(x,t)}}{\sqrt{g(x,0)}}.$$

Proof. Let $\{x_1, \cdots, x_n\}$ be a normal coordinate of N at p. Then, $\{x_1, \cdots, x_n, t\}$ is a local coordinate for $N \times (-\epsilon, \epsilon)$ near $(p, 0)$. Then for $t \in (-\epsilon, \epsilon)$, we have

$$dA_t = \sqrt{g(x,t)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

and

$$dA_0 = \sqrt{g(x,0)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Thus,

$$dA_t = \frac{\sqrt{g(x,t)}}{\sqrt{g(x,0)}}dA_0.$$

□

Theorem 2.3. *The first variation for the volume form at the point $(p, 0)$ is given by*

$$\frac{d}{dt}dA_t|_{(p,0)} = \left(\text{div}T^t + \langle T^n, \vec{H}\rangle\right) dA_0|_{(p,0)}.$$

Proof. Let $\{x_1, \cdots, x_n\}$ be a normal coordinates at $p \in N$. Define

$$J(x,t) := \frac{\sqrt{g(x,t)}}{\sqrt{g(x,0)}}.$$

Then

$$\frac{\partial}{\partial t}J(x,t)\bigg|_{t=0,x=p} = \frac{\partial}{\partial t}\left(\frac{\sqrt{g(x,t)}}{\sqrt{g(x,0)}}\right)\bigg|_{t=0,x=p} = \frac{1}{\sqrt{g(x,0)}}\frac{g'(x,t)}{2\sqrt{g(x,t)}}\bigg|_{t=0,x=p} = \frac{g'(p,0)}{2g(p,0)} = \frac{1}{2}g'(p,0)$$

because we chose normal coordinates. Note that

$$g(x,t) = \det(g_{ij}(x,t)) = \sum_{i=1}^{n} g_{1i}(x,t)C_{1i}(x,t),$$

where $C_{ij}(x,t)$ is a cofactor of a matrix $(g_{ij}(x,t))$. Then

$$g'(x,t) = \sum_{i=1}^{n} g'_{1i}(x,t)C_{1i}(x,t) + \sum_{i=1}^{n} g_{1i}(x,t)C'_{1i}(x,t).$$

Since $g_{1i}(p,0) = \delta_{1i}$ and $C_{1i}(p,0) = \delta_{1i}$, we have

$$g'(p,0) = g'_{11}(p,0) + C'_{11}(p,0).$$

We claim that $g'(p,0) = \sum_{i=1}^{n} g'_{ii}(p,0)$. To show it, we use mathematical induction on the dimension of N. For the base case $n = 1$, the equation

$$g'(p,0) = g'_{11}(p,0) + C'_{11}(p,0) = g'_{11}(p,0)$$

holds because $C_{11}(x,t) = 0$. Now, suppose

$$g'(p,0) = \sum_{i=1}^{k} g'_{ii}(p,0)$$

for any submanifold N of dimension k. Now assume that dimension of N is $k+1$. Then we can take a normal coordinates (U,ϕ) at $p \in N$ with local variables $\{x_1, \cdots, x_{k+1}\}$. Since $L = \{\phi(x) | x = (0, x_2, \cdots, x_{k+1}) \in U\}$ is k -dimensional submanifold of N, we have

$$\tilde{g}'(p,0) = \sum_{i=1}^{k} \tilde{g}'_{ii}(p,0),$$

where $\tilde{g}_{ij}(x,t)$ is a metric of L. Note that

$$C'_{11}(p,0) = \tilde{g}'(p,0) = \sum_{i=1}^{k} \tilde{g}'_{ii}(p,0) = \sum_{i=2}^{k+1} g'_{ii}(p,0).$$

Thus,

$$g'(p,0) = g'_{11}(p,0) + C'_{11}(p,0) = g'_{11}(p,0) + \sum_{i=2}^{k+1} g'_{ii}(p,0).$$

Hence

$$\frac{\partial}{\partial t} J(x,t)\bigg|_{t=0, x=p} = \frac{1}{2}\sum_{i=1}^{n} g'_{ii}(p,0).$$

On the other hand,

$$g'_{ii}(x,t) = \frac{\partial}{\partial t} g_{ii}(x,t) = T\langle e_i, e_i\rangle = 2\langle \nabla_T e_i, e_i\rangle = 2\langle \nabla_{e_i} T, e_i\rangle$$

because $\nabla_T e_i - \nabla_{e_i} T = [T, e_i] = 0$.

$$
\begin{aligned}
\frac{\partial}{\partial t} J(x,t)\Big|_{t=0,x=p} &= \sum_{i=1}^{n} \langle \nabla_{e_i} T, e_i \rangle \\
&= \sum_{i=1}^{n} \langle \nabla_{e_i} (T^t + T^n), e_i \rangle \\
&= \sum_{i=1}^{n} \langle \nabla_{e_i} T^t, e_i \rangle + \sum_{i=1}^{n} \langle \nabla_{e_i} T^n, e_i \rangle \\
&= \operatorname{div}(T^t) + \sum_{i=1}^{n} e_i \langle T^n, e_i \rangle - \sum_{i=1}^{n} \langle T^n, \nabla_{e_i} e_i \rangle \\
&= \operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle .
\end{aligned}
$$

Hence

$$
\frac{d}{dt} dA_t|_{(p,0)} = (\operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle) dA_0|_{(p,0)}.
$$

□

Remark. The right hand side of the above equation is not dependent of the choice of coordinates.

Corollary 2.4. *If T is a compactly supported variational vector field on N, then*

$$
\frac{d}{dt} A(N_t)\Big|_{t=0} = \int_N \langle T^n, \vec{H} \rangle .
$$

Proof.

$$
\int_{N_t} \frac{d}{dt} dA_t|_{p,0} = \frac{d}{dt}|_{t=0} \int_{N_t} dA_t = \frac{d}{dt}|_{t=0} A(N_t)
$$

$$
\int_{N_t} \frac{d}{dt} dA_t|_{p,0} = \int_{N_0} (\operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle) dA_0|_{(p,0)} = \int_N \langle T^n, \vec{H} \rangle dA_0
$$

□

Remark. Mean curvature of N is identically 0 if and only if N is a critical point of the area functional.

Definition 2.5. An immersed submanifold $N \to M$ is minimal if its mean curvature vector vanishes identically, i.e., $\vec{H} \equiv 0$.

Corollary 2.6. *Let N is a curve in M that is parametrized by arc-length with unit tangent vector e. Then the first variational formula for length can be written as*

$$
\frac{d}{dt} L(N_t)|_{t=0} = \langle T^t, e \rangle|_0^l - \int_{[0,l]} \langle T, \nabla_e e \rangle .
$$

Proof. Let $c : [0,l] \to M$ be a curve parametrized by arc-length. Then $N = c([0,l])$ is a submanifold of M. For each $p = c(x)$, a map $\phi : [0,l] \times (-\epsilon, \epsilon) \to N \times (-\epsilon, \epsilon)$ defined by $(c(x), t)$ is a local coordinate of $N \times (-\epsilon, \epsilon)$ at p. Let $e = d\phi(\frac{d}{dx})$ be a unit tangent vector of N_t. From the equation

$$
\frac{d}{dt} dA_t|_{(p,0)} = \left(\operatorname{div} T^t + \langle T^n, \vec{H} \rangle \right) dA_0|_{(p,0)},
$$

we have

$$
\frac{d}{dt} L(N_t)|_{t=0} = \int_{[0,l]} \operatorname{div} T^t + \int_{[0,l]} \langle T^n, \vec{H} \rangle .
$$

Note that

$$\text{div}T^t = \langle \nabla_e T^t, e\rangle = e\langle T^t, e\rangle - \langle T^t, \nabla_e e\rangle.$$

Since e is a unit vector, $\langle \nabla_e e, e\rangle = 0$, which implies that $(\nabla_e e)^t = 0$. Thus, $\langle T^t, \nabla_e e\rangle = 0$ and so

$$\int_{[0,l]} \text{div}T^t = \int_{[0,l]} e\langle T^t, e\rangle = \langle T^t, e\rangle|_0^l.$$

Also,

$$\int_{[0,l]} \langle T^n, \vec{H}\rangle = -\int_{[0,l]} \langle T^n, \nabla_e e\rangle = -\int_{[0,l]} \langle T, \nabla_e e\rangle.$$

In all,

$$\frac{d}{dt} L(N_t)|_{t=0} = \langle T^t, e\rangle|_0^l - \int_{[0,l]} \langle T, \nabla_e e\rangle.$$

□

2.2 Second Variational Formula for Area

General Setting Let (M, g) be a Riemannian manifold of dimension m and N be a submanifold of dimension $n < m$. Let $\phi : N \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$ be a two-parameter family of variations of N. For fixed t and s, denote by

$$N_{t,s} = \{\phi(x, t, s) | x \in N\}.$$

We assume that $N_{0,0} = N$. Then $N_{t,s}$ is an n-dimensional manifold.

We can induce the metric on $N_{t,s}$ from M. Take $p \in N$ and a **normal coordinates** $(U, \mathbf{x})$ of N at p. Let $\{x_1, \cdots, x_n\}$ be coordinate variables and $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ be coordinate vector fields around p. Then for fixed t and s,

$$x = (x_1, \cdots, x_n) \mapsto (\phi \circ \mathbf{x}(x), t, s)$$

is a coordinate chart of $N_{t,s}$ at $\phi(p, t, s)$ with coordinate variables $\{x_1, \cdots, x_n\}$.

Denote

$$e_i := d\phi\left(\frac{\partial}{\partial x_i}\right), \quad T = d\phi\left(\frac{\partial}{\partial t}\right), \quad S = d\phi\left(\frac{\partial}{\partial s}\right).$$

Then $\{e_1, \cdots, e_n\}$ is coordinate vector fields and we have a metric

$$g_{ij} = \langle e_i, e_j\rangle.$$

Moreover, $N \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ has coordinate variables $\{x_1, \cdots, x_n, t, s\}$ and coordinate vector fields $\{e_1, \cdots, e_n, T, S\}$.

Volume form and the first derivative Let's denote $dA_{t,s}$ be a volume form of $N_{t,s}$. Then

$$dA_{t,s} = \sqrt{g(x, t, s)} dx_1 \wedge \cdots \wedge dx_n,$$

where $g(x, t, s) = \det(g_{ij}(x, t, s))$. We denote $dA_{0,0} = dA$. Then,

$$dA_{t,s} = \sqrt{g(x, t, s)} dx_1 \wedge \cdots \wedge dx_n = \frac{\sqrt{g(x, t, s)}}{\sqrt{g(x, 0, 0)}} dA.$$

Let

$$J(x, t, s) := \frac{\sqrt{g(x, t, s)}}{\sqrt{g(x, 0, 0)}}.$$

Then

$$\frac{\partial J}{\partial t}(x,t,s) = \frac{1}{\sqrt{g(x,0,0)}} \cdot \frac{\frac{\partial}{\partial t} g(x,t,s)}{2\sqrt{g(x,t,s)}}. \tag{1}$$

By the fact

$$\frac{d}{dt}\det(A(t)) = \det A(t)\operatorname{tr}(A(t)^{-1}A'(t)),$$

we have

$$\frac{\partial}{\partial t} g(x,t,s) = g(x,t,s)\operatorname{tr}((g^{ij})(\dot{g}_{ij})) = g(x,t,s)\sum_{i=1}^{n}\sum_{k=1}^{n} g^{ik}\dot{g}_{ki}. \tag{2}$$

Note that

$$\dot{g}_{ki} = T\langle e_k, e_i\rangle = \langle \nabla_T e_k, e_i\rangle + \langle e_k, \nabla_T e_i\rangle = \langle \nabla_{e_k} T, e_i\rangle + \langle e_k, \nabla_{e_i} T\rangle \tag{3}$$

because T and e_i are coordinate vector fields. By eqs. (1) to (3), eq. (2), and eq. (3), we have

$$\begin{aligned}
\frac{\partial J}{\partial t}(x,t,s) &= \frac{g(x,t,s)\sum_{i=1}^{n}\sum_{k=1}^{n} g^{ik}\dot{g}_{ki}}{\sqrt{g(x,0,0)}\cdot 2\sqrt{g(x,t,s)}} \\
&= \frac{\sqrt{g(x,t,s)}}{2\sqrt{g(x,0,0)}}\sum_{i=1}^{n}\sum_{k=1}^{n} g^{ik}(\langle \nabla_{e_k} T, e_i\rangle + \langle \nabla_{e_i} T, e_k\rangle) \\
&= \frac{1}{2} J(x,t,s)\sum_{i,k=1}^{n} 2g^{ik}\langle \nabla_{e_i} T, e_k\rangle \\
&= J(x,t,s)\sum_{i,j=1}^{n} g^{ij}\langle \nabla_{e_i} T, e_j\rangle.
\end{aligned}$$

The second derivative Taking the derivative with respect to s, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial J}{\partial t} = \frac{\partial J}{\partial S}\sum_{i,j=1}^{n} g^{ij}\langle \nabla_{e_i} T, e_j\rangle &+ J(x,t,s)\sum_{i,j=1}^{n}\left(\frac{\partial}{\partial s} g^{ij}\right)\langle \nabla_{e_i} T, e_j\rangle \\
&+ J(x,t,s)\sum_{i,j=1}^{n} g^{ij}\frac{\partial}{\partial s}\langle \nabla_{e_i} T, e_j\rangle
\end{aligned}$$

- $\frac{\partial}{\partial s} g^{ij}$

  $$\begin{aligned}
  & g_{ik}g^{kj} = \delta_{ij}. \\
  \Rightarrow\ & (Sg_{ik})g^{kj} + g_{ik}(Sg^{kj}) = 0 \\
  \Rightarrow\ & g_{ik}(Sg^{kj}) = -(Sg_{ik})g^{kj} \\
  \Rightarrow\ & g^{li}g_{ik}(Sg^{kj}) = -g^{li}(Sg_{ik})g^{kj} \\
  \Rightarrow\ & \delta_{lk}(Sg^{kj}) = -g^{li}(Sg_{ik})g^{kj} \\
  \Rightarrow\ & Sg^{lj} = -g^{li}(Sg_{ik})g^{kj}.
  \end{aligned}$$

  Then at p,

  $$\begin{aligned}
  Sg^{lj} &= -g^{li}(Sg_{ik})g^{kj} \\
  &= -\delta^{li}(Sg_{ik})\delta^{kj} \\
  &= -Sg_{lj} \\
  &= -S\langle e_l, e_j\rangle \\
  &= -\langle \nabla_S e_l, e_j\rangle - \langle e_l, \nabla_S e_j\rangle \\
  &= -\langle \nabla_{e_l} S, e_j\rangle - \langle \nabla_{e_j} S, e_l\rangle.
  \end{aligned}$$

- $\sum g^{ij} \frac{\partial}{\partial s} \langle \nabla_{e_i} T, e_j \rangle$

  At p,

$$\begin{aligned}
\sum g^{ij} \frac{\partial}{\partial s} \langle \nabla_{e_i} T, e_j \rangle &= \delta_{ij} S \langle \nabla_{e_i} T, e_j \rangle \\
&= S \langle \nabla_{e_i} T, e_i \rangle \\
&= \langle \nabla_S \nabla_{e_i} T, e_i \rangle + \langle \nabla_{e_i} T, \nabla_S e_i \rangle \\
&= \langle R(S, e_i) T + \nabla_{e_i} \nabla_S T + \nabla_{[e_i, S]} T, e_i \rangle + \langle \nabla_{e_i} T, \nabla_S e_i \rangle \\
&= \langle R(S, e_i) T, e_i \rangle + \langle \nabla_{e_i} \nabla_S T, e_i \rangle + \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle.
\end{aligned}$$

Then at $(p, 0, 0)$,

$$\begin{aligned}
&\frac{\partial}{\partial s} \frac{\partial J}{\partial t} (p, 0, 0) \\
&= \frac{\partial J}{\partial s} \sum g^{ij} \langle \nabla_{e_i} T, e_j \rangle + J \sum_{i,j=1}^{n} \left(\frac{\partial}{\partial s} g^{ij} \right) \langle \nabla_{e_i} T, e_j \rangle + J \sum g^{ij} \frac{\partial}{\partial s} \langle \nabla_{e_i} T, e_j \rangle \\
&= J \sum g^{lk} \langle \nabla_{e_l} S, e_k \rangle \sum g^{ij} \langle \nabla_{e_i} T, e_j \rangle \\
&\quad + J \sum (-\langle \nabla_{e_i} S, e_j \rangle - \langle \nabla_{e_j} S, e_i \rangle) \langle \nabla_{e_i} T, e_j \rangle \\
&\quad + J \sum (\langle R(S, e_i) T, e_i \rangle + \langle \nabla_{e_i} \nabla_S T, e_i \rangle + \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle) \\
&= \sum_l \langle \nabla_{e_l} S, e_l \rangle \sum_i \langle \nabla_{e_i} T, e_i \rangle - \sum_{i,j} \langle \nabla_{e_i} S, e_j \rangle \langle \nabla_{e_i} T, e_j \rangle - \sum_{i,j} \langle \nabla_{e_j} S, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle \\
&\quad + \sum_i \langle R(S, e_i) T, e_i \rangle + \sum_i \langle \nabla_{e_i} \nabla_S T, e_i \rangle + \sum_i \langle \nabla_{e_i} T, \nabla_{e_i} S \rangle.
\end{aligned}$$

Second variational formula for a curve

Theorem 2.7. *Let M be an m-dimensional manifold. Let N be a curve parametrized by arc length in M with unit tangent vector given by e. Then the second variational formula for length is given by*

$$\left. \frac{\partial^2 L}{\partial s \partial t} \right|_{(s,t)=(0,0)} = \int_0^l -\langle \nabla_e S, e \rangle \langle \nabla_e T, e \rangle + \langle R(S, e) T, e \rangle + \langle \nabla_e \nabla_S T, e \rangle + \langle \nabla_e T, \nabla_e S \rangle$$

Proof. Since N is one dimensional and e is the unit tangent vector of N, we have

$$\begin{aligned}
\frac{\partial}{\partial s} \frac{\partial J}{\partial t} &= \langle \nabla_e S, e \rangle \langle \nabla_e T, e \rangle - \langle \nabla_e S, e \rangle \langle \nabla_e T, e \rangle - \langle \nabla_e S, e \rangle \langle \nabla_e T, e \rangle \\
&\quad + \langle R(S, e) T, e \rangle + \langle \nabla_e \nabla_S T, e \rangle + \langle \nabla_e T, \nabla_e S \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\left. \frac{\partial^2 L}{\partial s \partial t} \right|_{(s,t)=(0,0)} &= \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t)=(0,0)} \int_0^L dA_{s,t} \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t)=(0,0)} \int_0^L J(x, s, t) dA \\
&= \int_0^L \frac{\partial^2 J}{\partial s \partial t} (x, s, t) dA \\
&= \int_0^l -\langle \nabla_e S, e \rangle \langle \nabla_e T, e \rangle + \langle R(S, e) T, e \rangle + \langle \nabla_e \nabla_S T, e \rangle + \langle \nabla_e T, \nabla_e S \rangle
\end{aligned}$$

□

If we assume that the given curve is geodesic, then $\nabla_e e = 0$, so we have

$$\begin{aligned}\left.\frac{\partial^2 L}{\partial s \partial t}\right|_{(s,t)=(0,0)} &= \int_0^l -(e\langle S,e\rangle)(e\langle T,e\rangle) + \langle R(S,e)T,e\rangle + e\langle \nabla_S T, e\rangle + \langle \nabla_e T, \nabla_e S\rangle \\ &= [\langle \nabla_S T, e\rangle]_0^l + \int_0^l -(e\langle S,e\rangle)(e\langle T,e\rangle) + \langle R(S,e)T,e\rangle + \langle \nabla_e T, \nabla_e S\rangle.\end{aligned}$$

Second variational formula for a submanifold with the same direction

Theorem 2.8. *Let M be an m-dimensional manifold. Let N be an n-dimensional submanifold of M with $n < m$. Assume $T = S$, $T \perp N$, and T is compactly supported. Then we have*

$$\begin{aligned}\left.\frac{d^2}{dt^2} A(N_t)\right|_{t=0} &= \int \langle T, \vec{H}\rangle^2 - \int \sum \langle T, \vec{II}(e_i,e_j)\rangle^2 - \int \sum \langle R(e_i,T)T, e_i\rangle \\ &\quad + \int \langle (\nabla_T T)^\perp, \vec{H}\rangle + \int \sum_{i=1}^{n} \sum_{j=n+1}^{m} \langle \nabla_{e_i} T, e_j\rangle^2.\end{aligned}$$

Proof. Recall the following equation:

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial J}{\partial t} &= \sum_l \langle \nabla_{e_l} S, e_l\rangle \sum_i \langle \nabla_{e_i} T, e_i\rangle - \sum_{i,j} \langle \nabla_{e_i} S, e_j\rangle \langle \nabla_{e_i} T, e_j\rangle - \sum_{i,j} \langle \nabla_{e_j} S, e_i\rangle \langle \nabla_{e_i} T, e_j\rangle \\ &\quad + \sum_i \langle R(S,e_i)T, e_i\rangle + \sum_i \langle \nabla_{e_i} \nabla_S T, e_i\rangle + \sum_i \langle \nabla_{e_i} T, \nabla_{e_i} S\rangle.\end{aligned}$$

Since $S = T$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial J}{\partial t} &= \sum_l \langle \nabla_{e_l} T, e_l\rangle \sum_i \langle \nabla_{e_i} T, e_i\rangle - \sum_{i,j} \langle \nabla_{e_i} T, e_j\rangle \langle \nabla_{e_i} T, e_j\rangle - \sum_{i,j} \langle \nabla_{e_j} T, e_i\rangle \langle \nabla_{e_i} T, e_j\rangle \\ &\quad + \sum_i \langle R(T,e_i)T, e_i\rangle + \sum_i \langle \nabla_{e_i} \nabla_T T, e_i\rangle + \sum_i \langle \nabla_{e_i} T, \nabla_{e_i} T\rangle.\end{aligned}$$

- $(\sum_i \langle \nabla_{e_i} T, e_i\rangle)^2 = (\sum_i e_i\langle T, e_i\rangle - \langle T, \nabla_{e_i} e_i\rangle)^2 = \langle T, \vec{H}\rangle^2$.
- $\sum_{i,j} \langle \nabla_{e_i} T, e_j\rangle^2 = \sum_{i,j} \langle T, -\nabla_{e_i} e_j\rangle^2 = \sum_{i,j} \langle T, \vec{II}(e_i,e_j)\rangle^2$.
- $\langle \nabla_{e_j} T, e_i\rangle = e_j\langle T, e_i\rangle - \langle T, \nabla_{e_j} e_i\rangle = -\langle T, \nabla_{e_i} e_j\rangle = \langle \nabla_{e_i} T, e_j\rangle$.
- $\langle R(T,e_i)T, e_i\rangle = -\langle R(e_i,T)T, e_i\rangle$.
- $$\begin{aligned}\sum_i \langle \nabla_{e_i} \nabla_T T, e_i\rangle &= \sum_i \langle \nabla_{e_i} ((\nabla_T T)^T + (\nabla_T T)^\perp), e_i\rangle \\ &= \sum_i \langle \nabla_{e_i} (\nabla_T T)^T, e_i\rangle + \sum \langle \nabla_{e_i} (\nabla_T T)^\perp, e_i\rangle \\ &= \operatorname{div}(\nabla_T T)^T + \sum_i e_i \langle (\nabla_T T)^\perp, e_i\rangle - \sum_i \langle (\nabla_T T)^\perp, \nabla_{e_i} e_i\rangle \\ &= \operatorname{div}(\nabla_T T)^T + \langle (\nabla_T T)^\perp, \vec{H}\rangle\end{aligned}$$

- Let $\{e_{n+1}, \cdots, e_m\}$ denotes an orthonormal set of vectors normal to N in M. Then

$$\begin{aligned}\sum_i \langle \nabla_{e_i} T, \nabla_{e_i} T\rangle &= \sum_i \langle \nabla_{e_i} T, \sum_{j=1}^m \langle \nabla_{e_i} T, e_j\rangle e_j\rangle \\ &= \sum_{i,j=1}^n \langle \nabla_{e_i} T, \langle \nabla_{e_i} T, e_j\rangle e_j\rangle + \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, \langle \nabla_{e_i} T, e_j\rangle e_j\rangle \\ &= \sum_{i,j=1}^n \langle \nabla_{e_i} T, e_j\rangle^2 + \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, e_j\rangle^2.\end{aligned}$$

In all,

$$\begin{aligned}\frac{\partial^2 J}{\partial t^2}(x,t,s) =& \langle T, \vec{H}\rangle^2 - \sum \langle T, \vec{II}(e_i, e_j)\rangle^2 - \sum \langle \nabla_{e_i} T, e_j\rangle^2 \\ & - \sum \langle R(e_i, T)T, e_i\rangle + \operatorname{div}(\nabla_T T) + \langle (\nabla_T T)^\perp, \vec{H}\rangle \\ & + \sum_{i,j=1}^n \langle \nabla_{e_i} T, e_j\rangle^2 + \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, e_j\rangle^2.\end{aligned}$$

Thus, we now have

$$\begin{aligned}\frac{d^2}{dt^2} A(N_t)\Big|_{t=0} &= \frac{d^2}{dt^2} \int dA_t \\ &= \frac{d^2}{dt^2} \int J dA \Big|_{t=0} \\ &= \int \frac{d^2}{dt^2} J dA \Big|_{t=0} \\ &= \int \langle T, \vec{H}\rangle^2 - \sum \langle T, \vec{II}(e_i, e_j)\rangle^2 - \sum \langle R(e_i, T)T, e_i\rangle \\ &\quad + \int \operatorname{div}(\nabla_T T)^T + \langle (\nabla_T T)^\perp, \vec{H}\rangle + \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, e_j\rangle^2 \Big|_{t=0}.\end{aligned}$$

Divergence theorem implies

$$\int_N \operatorname{div}(\nabla_T T)^T = \int_{\partial N} \langle (\nabla_T T)^T, \nu\rangle = 0,$$

where ν is an outward unit normal vector of N. Thus we are done. $\square$

Definition 2.9. A minimally immersed submanifold N of M is **stable** if the second variation for area with respect to all compactly normal variations is non-negative.

Recall that N is minimal when $\vec{H} \equiv 0$. Thus, **stablity inequality** is

$$0 \le -\int \sum \langle T, \vec{II}(e_i, e_j)\rangle^2 - \int \sum \langle R(e_i, T)T, e_i\rangle + \int \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, e_j\rangle^2.$$

Codimension-1 minimal submanifold

Theorem 2.10. *Let M be an m-dimensional orientable manifold and N be an orientable codimension-1 minimal submanifold of M. Let $T = \psi e_m$, where ψ is a differentiable function on N and e_m is a unit normal vector field to N. Then the second variational formula is*

$$\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} = \int_N \left\{ -\psi^2 \sum_{i,j} h_{ij}^2 - \psi^2 Ric(e_m, e_m) + |\nabla\psi|^2 \right\},$$

where $\langle e_m, \vec{II}(e_i, e_j)\rangle = h_{ij}$ be the component of the second fundamental form. The stability inequality is

$$\int_N |\nabla\psi|^2 \geq \int_N \psi^2 h_{ij}^2 + \int_N \psi^2 Ric(e_m, e_m).$$

Proof.

$$\begin{aligned}\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} &= \int_N \left\{ -\sum_{i,j}\langle T, \vec{II}(e_i, e_j)\rangle^2 - Ric(T,T) + \sum_{i=1}^n \langle \nabla_{e_i} T, e_m\rangle^2 \right\} \\ &= \int_N \left\{ -\psi^2 \sum_{i,j}\langle e_m, \vec{II}(e_i, e_j)\rangle^2 - \psi^2 Ric(e_m,e_m) + \sum_{i=1}^n \langle \nabla_{e_i} T, e_m\rangle^2 \right\}.\end{aligned}$$

Note that

$$\langle \nabla_{e_i} T, e_m\rangle = \langle \nabla_{e_i}(\psi e_m), e_m\rangle = \langle \psi \nabla_{e_i} e_m, e_m\rangle + \langle e_i(\psi) e_m, e_m\rangle = e_i(\psi)$$

because e_m is a unit vector and

$$\langle \nabla_{e_i} e_m, e_m\rangle = \frac{1}{2} e_i \langle e_m, e_m\rangle = 0.$$

Then we have

$$\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} = \int_N \left\{ -\psi^2 \sum_{i,j} h_{ij}^2 - \psi^2 Ric(e_m, e_m) + |\nabla\psi|^2 \right\}$$

□

Geodesic balls of radius r Let M be an m-dimensional oriented manifold and N be the n-dimensional geodesic balls of radius r in M. We want to control the growth of the volume of N. Let $T = e_m$ and $\nabla_{e_m} e_m = 0$. Denote $\vec{H} = H e_m$. Then the first variational formula is

$$\frac{d}{dt}A(N_t)\bigg|_{t=0} = \int_N \langle T^\perp, \vec{H}\rangle = \int_N H$$

because $\mathrm{div} T^T$ term vanishes. The second variational formula is

$$\begin{aligned}\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} &= \int \langle T, \vec{H}\rangle^2 - \int \sum \langle T, \vec{II}(e_i, e_j)\rangle^2 - \int \sum \langle R(e_i, T)T, e_i\rangle \\ &\quad + \int \langle (\nabla_T T)^\perp, \vec{H}\rangle + \int \sum_{i=1}^n \sum_{j=n+1}^m \langle \nabla_{e_i} T, e_j\rangle^2 \\ &= \int \langle e_m, H e_m\rangle^2 - \int \sum h_{ij}^2 - \int Ric(e_m, e_m) + \int \sum_{i=1}^n \langle \nabla_{e_i} e_m, e_m\rangle^2 \\ &= \int H^2 - \int \sum h_{ij}^2 - \int Ric(e_m, e_m).\end{aligned}$$

3 Stable minimal submanifold and scalar curvature

3.1 Geometry of 2nd variation

Definition 3.1. We say a minimal submanifold $N \subset M$ is stable if $\frac{d}{dt}Area(N_t) \geq 0$. Equivalently, the minimal submanifold is a local minimum of area functional.

The scalar curvature has topological implications on stable minimal sub-manifolds, especially in low dimension:

Theorem 3.2 (Schoen-Yau)**.** *Let M^3 be a 3 dimensional compact oriented manifold with positive scalar curvature (PSC). Then M has no compact immersed stable minimal surface of positive genus.*

Before we start the proof, we recall several tools that we are going to use:

Recall. Stability condition of minimal hypersurface $N^n \subset M^{n+1}$: if e_{n+1} is the unit normal vector to N in M, $T = \psi e_{n+1}$ is the variational field with $\psi \in C_c^\infty(N)$. Then N is stable iff

$$\int_N |\nabla\psi|^2 - (\mathrm{Ric}(e_{n+1}) + |II|^2)\psi^2 \geq 0$$

And a lemma from Gauss' equation for hypersurface:

Lemma 3.3. *For a minimal hypersurface $N \subset M$, one has*

$$\mathrm{Ric}(e_{n+1}) = \frac{1}{2}(S_M - S_N - |II|^2)$$

where S_M is the scalar curvature of M.

Proof. Let $e_1, ..., e_n, e_{n+1}$ be a local orthonormal frame at $p \in N$ with e_{n+1} normal to N. Then

$$\begin{aligned} S_M &= \sum_{i,j=1}^{n+1} R^M(e_i, e_j, e_j, e_i) \\ &= 2\mathrm{Ric}(e_{n+1}) + \sum_{i,j=1}^{n} R^M(e_i, e_j, e_j, e_i) \\ &= 2\mathrm{Ric}(e_{n+1}) + \sum_{i,j=1}^{n} R^N(e_i, e_j, e_j, e_i) - II(e_i, e_i)II(e_j, e_j) + (II(e_i, e_j)^2) \\ &= 2\mathrm{Ric}(e_{n+1}) + S_N - H^2 + |II|^2 \\ &= 2\mathrm{Ric}(e_{n+1}) + S_N|II|^2 \end{aligned}$$

where $H = 0$ is the mean curvature of N. Rearranging the equation, the lemma is proved. □

With the lemma, the stability condition becomes

$$\int_N |\nabla\psi|^2 - \frac{1}{2}(S_M - S_N - |II|^2)\psi^2 \geq 0$$

With all these preparations, we are ready to prove Schoen-Yau's theorem above:

Proof. For sake of contradiction, suppose there exists a stable minimal surface $N^2 \subset M^3$ with positive genus. Stability condition implies

$$\int_N |\nabla\psi|^2 - \frac{1}{2}(S_M - S_N - |II|^2)\psi^2 \geq 0 \textit{ for any } \psi \in C_c^\infty(N)$$

Take $\psi = 1$ to get

$$\int_N (S_M - S_N - |II|^2) \leq 0$$

Now with the PSC condition, $S_M > 0$. As for two dimensional surface, $S_N = 2K$ where K is the Gaussian curvature of N. Hence by Gauss-Bonnet, $\int_N S_N = 2 \cdot 2\pi(2-2g) \leq 0$, since N has positive genus g. Hence the second variation is negative, a contraction to the stability condition. $\square$

Some question that one may ask:

1. What about higher dimensional case?
2. Existence of stable minimal hypersurface?
3. What if N is not compact?

To answer Q2, we usually try to minimize area subject to some topological constraints. For example, given a curve γ, we want to minimize its arc length in a non-trivial free homotopy class $\alpha \in [S^1, M]$. Existence results show that α can be represented by a closed geodesic (hence locally minimizing) $\Rightarrow$ stable.
We have the following theorem:

Theorem 3.4 (Schoen-Yau)**.** *Let M be a compact Riemannian manifold whose fundamental group $\pi_1(M)$ contains a free ablelian group of rank 2. Then there exists a branched minimal immersion $f : T^2 \to M$, whose image of $\pi_1(T^2)$ under the induced map f_* is a free abelian group of rank 2. Moreover, f minimizes area among all such maps (hence is stable).*

Remark. In dim $M = 3$, Gullien-Ossermann: No branch point.

Corollary 3.5. *Let M^3 be a compact oriented manifold. If $\pi_1(M)$ contains a free ablelian group of rank 2, then M cannot admit a metric of PSC.*

Proof. Combine the two theorems by Schoen and Yau. $\square$

As applications of the theorems above, we consider some examples here: T^3 contains a a free ablelian group of rank 2 in its fundamental group, so it cannot carry a metric with PSC. In fact this is true for T^n. This implies that R^3 cannot have a PSC metric which is Euclidean outside a ball of large radius.
We now go back to Q1: What about higher dimensions? We start from the stability condition:

$$\int_N |\nabla\psi|^2 - \frac{1}{2}(S_M - S_N - |II|^2)\psi^2 \geq 0 \textit{ for any } \psi \in C_c^\infty(N)$$

$\Longrightarrow$

$$\int_N |\nabla\psi|^2 - \frac{1}{2}(S_M - S_N)\psi^2 \geq \frac{1}{2}\int_N |II|^2\psi^2 \geq 0$$

Assume $S_M > 0$ and N is compact. Then

$$\int_N |\nabla\psi|^2 + \frac{1}{2}S_N\psi^2 \geq \frac{1}{2}\int_N S_M\psi^2 \geq c_0\int_N \psi^2$$

where $c_0 = \frac{1}{2}\min_N S_M > 0$. $\Longrightarrow$ the first eigenvalue of the elliptic operator $\Delta_N + \frac{1}{2}S_N$ is positive. We have the following lemma:

Lemma 3.6. *Let $L = \Delta_N + \frac{n-2}{4(n-1)}S_N$ be the conformal laplacian. If $N^n \subset M^{n+1}$ is a compact, oriented stable minimal hypersurface and $S_M > 0$. Then the first eigenvalue of L is positive.*

Proof. As before we have

$$\int_N |\nabla\psi|^2 + \frac{1}{2}S_N\psi^2 \geq c_0 \int_N \psi^2$$

Now

$$\begin{aligned}
\text{LHS} &= \frac{2(n-1)}{n-2}\int_N \frac{n-2}{2(n-1)}|\nabla\psi|^2 + \frac{n-2}{4(n-1)}S_N\psi^2 \\
&< \frac{2(n-1)}{n-2}\int_N |\nabla\psi|^2 + \frac{n-2}{4(n-1)}S_N\psi^2 \\
&= \frac{2(n-1)}{n-2}(L\psi,\psi)_{L^2}
\end{aligned}$$

And note that LHS $\geq c_0||\psi||^2_{L^2}$, hence

$$\lambda_1(L) \geq \frac{n-2}{2(n-1)}c_0 > 0$$

This completes the proof of the lemma. □

Remark. Why do we call L in the lemma "Conformal Laplacian"? Consider the conformal transformation $\tilde{g} = \psi^{\frac{4}{n-2}}g, \psi \in C^\infty(N)$. Then

$$S_{\tilde{g}} = \psi^{-\frac{n+2}{n-2}}(4\frac{n-1}{n-2}\Delta\psi + S_g\psi) = \frac{4(n-1)}{n-2}\psi^{-\frac{n+2}{n-2}}L\psi$$

by the conformal change formulae.

With this lemma in mind, we could prove that a compact orientable stable minimal hypersurface of a PSC manifold has a metric (conformal to the induced metric) with positive scalar curvature:

Theorem 3.7. *Let M^{n+1} be an oriented Riemannian manifold with PSC. Let $N^n \subset M^{n+1}$ be a compact oriented stable minimal hypersurface. Then N admits a PSC-metric.*

Proof. Let ψ be the first eigenfunction of the conformal laplacian L. WLOG we could assume $\psi > 0$. Let $\tilde{g}_N = \psi^{\frac{4}{n-2}}g_N$ be a metric on N conformal to the induced metric. Then

$$\begin{aligned}
\tilde{S} &= \frac{4(n-1)}{n-2}\psi^{-\frac{n+2}{n-2}}L\psi \\
&= \frac{4(n-1)}{n-2}\psi^{-\frac{n+2}{n-2}}\lambda_1\psi \\
&= \lambda_1\frac{4(n-1)}{n-2}\psi^{-\frac{4}{n-2}} > 0
\end{aligned}$$

□

3.2 GMT Approach to Stable Minimal Hypersurface

Now we turn to the question of existence of stable minimal hypersurface in a PSC manifold M. The general idea is that we minimize some functional subject to some topological constraint. Two common choices of the functional are area functional and energy functional. The first choice requires some geometric measure theory (GMT) techniques and gives us minimal surface directly. The second choice leads to harmonic maps. We now focus on the first choice. Here we need to borrow a big result from GMT:

Lemma 3.8. *Let M^n be a closed oriented manifold with PSC, and $\alpha \in H_{n-1}(M, \mathbb{Z})$ a nontrivial homology class. Suppose $3 \leq dim\ M \leq 7$. Then*

$$\alpha = [N_1] + [N_2] + \cdots [N_k]$$

where each N_i is an embedded oriented stable minimal hypersurface and they are disjoint from each other.

Remark. The dimensional restriction $3 \leq \dim M \leq 7$ is for obtaining regularity of N_i. And α has codimension $1 \Rightarrow$ nice regularity. If dimension is higher than 7, you may encounter singularities in N_i.

Here is an important example that we state as a lemma:

Lemma 3.9. *There is no PSC metric on T^3.*

Proof. We know that

$$\omega_1 = dx, \omega_2 = dy, \omega_3 = dz \in H^1_{dR}(T^3)$$

are closed but not exact 1-forms on T^3. By Poincaré duality, each ω_i is dual to some nontrivial $\alpha_i \in H_2(M, \mathbb{Z})$. If, for simplicity, $\alpha_1 = [N_1]$ represented by only a single minimal surface, we could use lemma 3.8 to get N_1 is stable, and therefore by theorem 3.7 admits a metric with PSC. Then theorem 3.2 tells us that $N_1 \cong \mathbb{S}^2$. Moreover, α_1 is the Poincaré dual of ω_1 means

$$\int_M \omega_1 \wedge \eta = \int_{N_1} \eta, \quad \forall \eta \in H^2_{dR}(T^3)$$

In particular we take $\eta = \omega_2 \wedge \omega_3$:

$$1 = \int_M \omega_1 \wedge \omega_2 \wedge \omega_3 = \int_N \omega_2 \wedge \omega_3$$

So $\omega_2|_{N_1} \neq 0 \in H^1_{dR}(N_1)$.(Otherwise, $\omega_2|_{N_1} = df$) and

$$\int_N \omega_2 \wedge \omega_3 = \int_N df \wedge \omega_3 = \int_N d(f\omega_3) = 0$$

as $N_1 \cong \mathbb{S}^2$ has no boundary. But $H^1_{dR}(N_1) = H^1_{dR}(\mathbb{S}^2) = 0$, contradiction. Conclusion: we cannot have PSC on T^3. □

Remark. If in the proof above α has multiple N_i:

$$\alpha = [N_1] + [N_2] + \cdots [N_k]$$

Then

$$1 = \sum_{i=1}^{k} \int_{N_i} \eta$$

so there exists at least one N_i with non zero integral of η. The contradiction still follows.

We can now state the main theorem:

Theorem 3.10 (Schoen-Yau)**.** *Let $M^{n+1}(3 \leq n+1 \leq 7)$ be a closed oriented manifold, $\omega_1 \cdots \omega_{n+1} \in H^1_{dR}(M)$ nontrivial 1-forms such that $\omega_1 \wedge \cdots \wedge \omega_{n+1} \neq 0 \in H^{n+1}_{dR}(M)$ Then M has no PSC-metric.*

Proof. The proof of theorem 3.10 is basically an induction using lemma 3.9. Take $\alpha_1, \cdots, \alpha_{n+1}$ non trivial homology class in $H_n(M)$ with Poincaré duals $\omega_1, \cdots \omega_{n+1}$ nontrivial in $H^1_{dR}(M)$. Assume by contradiction that M has a PSC-metric, By lemma 3.8, one of the components of a representative of α_1, say N_1 has non zero integral:

$$\int_{N_1} \omega_2\omega_2 \wedge \cdots \wedge \omega_{n+1} \neq 0$$

theorem 3.7 implies that N_1 has a PSC-metric. A similar argument as in lemma 3.9 shows that $\omega_2|_{N_1}, \cdots, \omega_2|_{N_1} \neq 0 \in H^1_{dR}(N_1)$. Since

$$\int_{N_1} \omega_2\omega_2 \wedge \cdots \wedge \omega_{n+1} \neq 0$$

we can iterate the above argument on N_1 to reduce the dimension until theorem 3.2 works and a contradiction similar to lemma 3.9 follows. □

Two corollaries are the following:

Corollary 3.11. *$T^n(3 \leq n \leq 7)$ has no PSC-metric.*

Proof. Take $\omega_i = dx_i$, then apply the above theorem. □

Corollary 3.12. *If $M^n(3 \leq n \leq 7)$ has a smooth map*

$$F : M^n \longrightarrow T^n$$

of non-zero degree, then M cannot have a PSC-metric.

Proof. We pullback the 1-forms from T^n:

$$\omega_i = F^*(dx_i)$$

then

$$\begin{aligned}
\int_M \omega_1 \wedge \cdots \wedge \omega_n &= \int_M F^*(dx_1 \wedge \cdots \wedge dx_n) \\
&= \deg(f) \int_{T^n} dx_1 \wedge \cdots \wedge dx_n \\
&\neq 0
\end{aligned}$$

Now apply the main theorem above. □

In particular, ifn $M_0^n(3 \leq n \leq 7)$ is any closed manifold, then there exists a degree 1 map

$$F : M_0^n \# T^n \longrightarrow T^n$$

that collapses M_0 to a point in T^n and maps T^n to itself. So $M_0^n \# T^n$ cannot have PSC-metric, which, as we will show in the future, implies Positive Mass Theorem (PMT).

4 Positive Scalar Curvature and General Relativity

Let (N, g) be a space-time, i. e. a Lorentzian manifold satisfying the following Einstein field equation

$$Ric_N - \frac{1}{2} S_N g = T.$$

Here T is the stress-energy tensor.

Let $M^n \subset N^{n+1}$ be a (space-like) submanifold, $e_0, e_1, ..., e_n$ be a local frame on M, s.t. $e_0 \perp TM$. Then the Einstein field equation in the normal direction yields

$$\frac{1}{2}S_M + (trA)^2 - |A|^2 = \mu. \tag{4}$$

Here $\mu = T_{00}$ is the mass density (as observed by an observer traveling along e_0 direction).

Similarly, taking the $(0j)$ components (and using the Codazzi equation), we have

$$div[A - (trA)g] = J, \tag{5}$$

where $J_j = -T_{0j}$ is the observed momentum 1-form.

The equations (4) and (5) together are called the constraint equations. They consitute necessary conditions for a Riemannian manifold to be spatial slice of a space-time (and is important in the initial value formulation of GR, i.e, the dynamical viewpoint of the space-time).

A particular important case is the time-symmetric slices, i. e. $A \equiv 0$(which means M is a total geodesic submanifold). In this case, (4) tells us that

$$\frac{1}{2}S_M = \mu \geq 0.$$

Hence the nonnegativity is imposed by physical intuition.

Thus, a Riemannian manifold occuring as the time-symmetric slice of a space-time must have nonnegative scalar curvature.

5 Positive Mass Theorem (PMT)

In physics, Space-time describing isolated physical system should approach Minkovski space at ∞. Correspondingly, its spatial slices are the so-called asymptotically flat (AF) or asymptotically Euclidean (AE) manifold.

Definition 5.1. A complete noncompact Riemannian manifold (M^n, g) is called AF if, outside a compact subset $K \subset M$, $M - K$ is diffeomorphic to $\mathbb{R}^n - B_1(0)$. Moreover, if $x_1, ..., x_n$ are the pull-back of Euclidean coordinates via the diffeomorphism, then

$$g_{ij} = \delta_{ij} + O(|x|^{-\delta}), \delta > \frac{n-2}{2},$$

$$\partial g_{ij} = O(|x|^{-\delta-1}),$$
$$\partial^2 g_{ij} = O(|x|^{-\delta-2}),$$
$$S_g = O(|x|^{-q}), q > n.$$

Example 5.2 (Schwarzschild)**.** On $\mathbb{R}^n - \{0\}$, $g_{ij} = (1 + \frac{m}{2|x|^{n-2}})^{\frac{4}{n-2}}\delta_{ij}, m \geq 0$, is AF with $\delta = n-2$.

Definition 5.3 (ADM mass)**.** Let (M, g) be AF, then the ADM mass m_{ADM} is defined by

$$m_{ADM} = \frac{1}{4(n-1)w_{n-1}} \lim_{R\to\infty} \int_{|x|=R} (\partial_i g_{ij} - \partial_j g_{ii})\nu^i dS.$$

Here ν is the outer normal of the sphere.

Remark. The limit above exists because of the asymptotic conditions.

Theorem 5.1 (Positive Mass Theorem)**.** *If (M^n, g) is AF, then $S_g \geq 0$ implies $m_{ADM} \geq 0$. And equality holds iff $(M^n, g) = (\mathbb{R}^n, \delta)$.*

Remark. Schoen and Yau proved PMT by minimal surface techniques in the case of $n \leq 7$ (c.f. [**?**]). For higher dimension, Schoen and Yau recently give a proof of PMT by dealing with singularities of minimal surfaces (c.f. [**?**]). Witten also gives a proof of PMT for spin manifold (c.f. [**?**]).

Corollary 5.4 (Geroch Conjecture)**.** *$\mathbb{R}^n$ has no compact perturbation maintaining $S \geq 0$.*

As we mensioned in Section **??**, T^4 has no PSC. In fact,

Theorem 5.2. *For any closed manifold M^n, $M^n \# T^n$ has no PSC, if either M is spin or $n \leq 7$.*

By the following theorem, we can see how Theorem 5.2 and Theorem 5.1 are related.

Theorem 5.3 (Compactification Theorem)**.** *Supposed that for all closed manifold M^n, $M^n \# T^n$ has no PSC, then PMT holds.*

6 Dirac Operator, Spin Structures, Lichnerowicz Formula and Its Application

6.1 Motivation

According to Einstein's (special) relativity, a free particle of mass m in $\mathbb{R}^3$ with momentum vector $p = (p_1, p_2, p_3)$ has energy

$$E = c\sqrt{m^2c^2 + p^2} = c\sqrt{m^2c^2 + p_1^2 + p_2^2 + p_3^2}.$$

For simplicity, we assume that $c = 1$. Passing to quantum mechanics, one replaces E by the operator $i\frac{\partial}{\partial t}$, and p_j by $-i\frac{\partial}{\partial x_j}$. Therefore the particle now is described by a state function $\Psi(t, x)$ satisfying the equation

$$i\frac{\partial \Psi}{\partial t} = \sqrt{m^2 + \Delta}\Psi.$$

Here the Laplacian

$$\Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}.$$

This motivates Dirac to look for a (Lorentz invariant) square root of Δ. In other words, Dirac looks for a first order differential operator with constant coefficients

$$D = \gamma_j \frac{\partial}{\partial x_j} + m\gamma_0$$

such that $D^2 = m^2 + \Delta$. It follows that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \quad \text{if } 0 \leq i \neq j \leq 3; \quad \gamma_0^2 = 1 \text{ and } \gamma_i^2 = -1 \quad \text{for } i = 1, 2, 3.$$

Dirac realized that, to have solutions, the coefficients γ_i will have to be complex matrices.

6.2 Clifford Algebra

To generalized Dirac operator on higher dimensional manifolds, we introduce Clifford algebra.

Definition 6.1 (Clliford algebra)**.** Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional Euclidean space with an orthonormal basis $\{e_i\}_{i=1}^n$. The Clifford algebra $Cl(V)$ (or denoted by Cl_n) is the real algebra generated by $1, e_1, \cdots, e_n$ subject only to the relations

$$e_ie_j + e_je_i = -2\delta_{ij}.$$

It is clear that

$$1, e_1, \cdots, e_{n_1} e_1 e_2, \cdots, e_{i_1} e_{i_2} \cdots e_{i_k} \left(i_1 < i_2 < \cdots < i_k\right), \cdots, e_1 \cdots e_n$$

is a vector space basis for Cl_n. Hence $Cl_n \cong \Lambda^* V$ as vector spaces (they are actually isomorphic as Clifford module).

Example 6.2. One can see esaily that $Cl_1 \equiv \mathbb{C}$, where e_1 corresponds to i. $\mathrm{Cl}_2 \equiv \mathrm{H}$, the quaternions, and the basis vectors $e_1, e_2, e_1 e_2$ correspond to I, J, K.

Definition 6.3 (Complexification of Clifford Algebra)**.** We consider the complexification of the Clifford algebra

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}.$$

Example 6.4. First, one can see essily that

$$\begin{aligned} \mathbb{C}l_1 = Cl_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{C} \oplus \mathbb{C}, \\ \mathbb{C}l_2 = Cl_2 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \equiv End(\mathbb{C}^2). \end{aligned}$$

In fact, one has

Theorem 6.5. *One has the mod 2 periodicity*

$$\mathbb{C}l_n = \begin{cases} \mathrm{End}\left(\mathbb{C}^{2^{n/2}}\right) & \textit{if } n \textit{ is even;} \\ \mathrm{End}\left(\mathbb{C}^{2^{(n-1)/2}}\right) \oplus \mathrm{End}\left(\mathbb{C}^{2^{(n-1)/2}}\right) & \textit{if } n \textit{ is odd}. \end{cases}$$

Definition 6.6. A Clifford module (M, c) consists of a $\mathbb{C}$-vector space M and a morphism $c : \mathbb{C}l_n \to End(M)$. Then $\mathbb{C}l_n$ acts on M as matrix multiplication via c.

Example 6.7. The exterior algebra $\Lambda^* V \otimes_{\mathbb{R}} \mathbb{C}$ is a Clifford module, the Clifford action is given by

$$c(e_i) w = e_i \wedge w - \iota_{e_i} w,$$

where ι is the interior product.

By the mod 2 periodicity, one can see that, when n is even, $\mathbb{C}l_n$ has a canonical $2^{n/2}$-dimensional module, denoted by (Δ_n, c), whose Clifford action c is given by the matrix multiplication; when n is odd, $\mathbb{C}l_n$ has two canonical $2^{n/2}$-dimensional module, denoted by $(\Delta_n^i, c), i = 0, 1$, whose Clifford action c is given by the matrix multiplication of i-th components.

6.2.1 Dirac operator on $\mathbb{R}^n$

Now, we are in a position to talk about Dirac operator on $\mathbb{R}^n$. Given a Clifford module (M, c), the Dirac operator $D := \sum_{i=1}^n c(e_i) \partial_i$ is a first order differential on M-valued function on $\mathbb{R}^n$. Moreover, one can check easily that $D^2 = \Delta$.

6.3 Dirac operator

Let (X^n, g) be a closed Riemannian manifold of dimension n, locally, for any Clifford module M, the construction in Section 6.2.1 could be done. The problem is that one can't glue the locally construction usually, and there are some topological obstruction. However, if (X, g) is spin, such construction could be done.

Definition 6.8. We say a Riemannian manifold (X, g) is spin if $w_0(X)$ and $w_1(X)$ vanish, where w_0 and w_1 are Stiefel-Whitney of tangent bundle.

Remark. $w_0 = 0$ iff M is orientable.

If (X, g) is spin, then

Theorem 6.9. *There exists a Hermitian vector bundle $(S \to X, \langle \cdot, \cdot \rangle)$, called spinor bundle, such that*

1. *S has a unitary connection ∇^S.*
2. *together with a Clifford action $c : \Gamma(T^*X) \times \Gamma(S) \to \Gamma(S)$ satisfying*
   - *(Leibniz's rule)$\nabla^S(c(v)s) = c(\nabla^{LC}v)s + c(v)\nabla^S s$ for all $v \in \Gamma(T^*M), s \in \Gamma(S)$, where ∇^{LC} is the Levi-Civita connection.*
   - *If $g(v, v) = 1$, then $\langle c(v)s_1, c(v)s_2 \rangle = \langle s_1, s_2 \rangle$ for all $v \in \Gamma(T^*M), s_1, s_2 \in \Gamma(S)$.*

Moreover, suppose locally $\nabla^{LC} e_i = \sum_j w_{ij} e_j$, then connection ∇^S could be given by

$$\nabla^S = d + \sum_{i,j} \frac{w_{ij}}{4} c(e_i)c(e_j). \tag{6}$$

Remark. When $M = \mathbb{R}^n$, $S := \mathbb{R}^n \times \Delta_n$.

Example 6.10.

- $T^n, \mathbb{R}^n$, any Lie group G and any 3 dimensional orientable manifolds are spin, since their tangent bundle are trivial.
- All orientable surfaces are spin.
- A complex manifold X is spin iff $c_1(X) \equiv 0 (\text{mod} 2)$.
- $\mathbb{RP}^n$ is spin iff $n \equiv 3 \bmod 4; \mathbb{CP}^n$ is spin iff n odd ($n \equiv 1 \bmod 2$); HP^n is always spin.
- Since $\{w_i\}$ are homotopy invariants, hence if X and Y are homotopic equivalent, then X is spin iff Y is spin.

6.4 Lichnerowicz Formula

Definition 6.11. If (X, g) is spin, and let $S \to X$ be the spinor bundle, then the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is defined by

$$D := \sum_i c(e_i) \nabla^S_{e_i},$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame of T^*X.

Theorem 6.12. $D^2 = \Delta + \frac{k}{4}$, *where k is the scalar curvature on X.*

Proof. Assume that at $p \in X$, $\nabla^{LC} e_i = 0$, then by a straightforward computation,

$$
\begin{aligned}
D^2 :&= \sum_{i,j} c(e_i)\nabla^S_{e_i} c(e_j)\nabla^S_{e_j} \\
&= \sum_{i} c(e_i)\nabla^S_{e_i} c(e_i)\nabla^S_{e_i} + \sum_{i\neq j} c(e_i)\nabla^S_{e_i} c(e_j)\nabla^S_{e_j} \\
&= \sum_{i} c(e_i)c(e_i)\nabla^S_{e_i}\nabla^S_{e_i} + \sum_{i\neq j} c(e_i)c(e_j)\nabla^S_{e_i}\nabla^S_{e_j} \text{ (By Leibniz's rule and } \nabla^{LC} e_i = 0) \\
&= -\sum_{i} \nabla^S_{e_i}\nabla^S_{e_i} + \sum_{i<j} c(e_i)c(e_j)(\nabla^S_{e_i}\nabla^S_{e_j} - \nabla^S_{e_j}\nabla^S_{e_i}) \text{ (Since } c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij}) \\
&= \Delta + \sum_{i<j} c(e_i)c(e_j)R^S(e_i, e_j) \\
&= \Delta + \frac{1}{8} R_{ijkl}\, c\left(e_i\right) c\left(e_j\right) c\left(e_k\right) c\left(e_l\right) \text{ (By (6))} \\
&= \Delta + \frac{1}{8}\sum_l \left[\frac{1}{3}\sum_{i,j,k} \left(R_{ijkl} + R_{jkil} + R_{kijl}\right) c\left(e_i\right) c\left(e_j\right) c\left(e_k\right) \right. \\
&\left. + \sum_{i,j} R_{ijil}\, c\left(e_i\right) c\left(e_j\right) c\left(e_i\right) + \sum_{i,j} R_{ijjl}\, c\left(e_i\right) c\left(e_j\right) c\left(e_j\right) \right] c\left(e_l\right) \\
&= \Delta + \frac{1}{4} R_{ijil}\, c\left(e_j\right) c\left(e_l\right) \text{ (By Bianchi identity)} \\
&= \Delta - \frac{1}{4}\operatorname{Ric}\left(e_j, e_l\right) c\left(e_j\right) c\left(e_l\right) \\
&= \Delta + \frac{1}{4}\operatorname{Ric}\left(e_j, e_l\right) \delta_{jl} \\
&= \Delta + \frac{k}{4}
\end{aligned}
$$

□

When (X, g) admits PSC, D^2 is a strictly positive operator, hence by Atiyah-Singer index theorem

Theorem 6.13. *Let (X, g) be closed and spin, then if (X, g) admits PSC, then $\hat{A}$-genus vanishes. Here $\hat{A}$-genus is given by*

$$\int_X \hat{A}(M).$$

Remark. The inverse is not ture. In fact, by a more refined argument (we introduce the notion of enlargibility), one can show that T^n can't admit a metric of PSC, but its $\hat{A}$-genus vanishes. Indeed, one can prove that if X is closed and spin, $X\# T^n$ cannot admit a metric of PSC (we prove this before in the lower dimension using the minimal surface technique without assuming the spin condition).

7 Witten's proof of positive mass theorem

Recall that we stated the positive mass theorem 5.1: If (M^n, g) is AF, then $S_g \geq 0$ implies $m_{ADM} \geq 0$. And equality holds iff $(M^n, g) = (\mathbb{R}^n, \delta)$.

Assume that M is spin. We shall present Witten's proof of the theorem. The strategy is to use Lichnerowicz formula 10.4

$$D^2 = \nabla^* \nabla + \frac{S}{4}.$$

If M is compact, we have seen that $S > 0$ implies $\hat{A}(M) = 0$, using integration by parts. Now for asymptotically flat manifolds, we need to deal with boundary contribution, which turns out to be the ADM mass.

Definition 7.1. A spinor ϕ is called *harnomic* if $D\phi = 0$. A spinor ϕ is called *parallel* if $\nabla^S \phi = 0$.

Remark. A parallel spinor is harmonic since $D = c(e_i)\nabla^S_{e_i}$.

Example 7.2. Let $M = \mathbb{R}^n$ and $g = g_{Euc}$. Then the spinor bundle over M is trivial and $D = \gamma_i \frac{\partial}{\partial x_i}$. The parallel spinors are just constant spinors.

Outline of the proof —

1. For any parallel spinor ϕ_0 on $\mathbb{R}^n$, construct a harmonic spinor ϕ on M such that $\phi \to \phi_0$ at infinity.

2. Apply the Lichnerowicz formula and integration by parts to get

$$m_{ADM} = \int_M (|\nabla^s \phi|^2 + \frac{S}{4}|\phi|^2) dvol.$$

We first prove a weaker theorem to get familiarized with the ideas.

Theorem 7.3. *Assume that (M^n, g) is AF with $n \geq 3$, $\tau > \frac{n-2}{2}$. If $\mathrm{Ric}(g) \geq 0$ then $m_{ADM} \geq 0$. The equality holds iff (M, g) is isometric to $(\mathbb{R}^n, g_{Euc})$.*

Remark. The theorem is actually too weak. If $\mathrm{Ric}(g) \geq 0$, then we can apply volume comparison to see that $\frac{|B_r(p)|}{\omega_n r^n}$ is decreasing and has limit 1 when $r \to 0$. The asymptotic flatness implies that it has limit 1 when $r \to \infty$ as well. Hence, the rigidity of volume comparison implies that M is isometric to $(\mathbb{R}^n, g_{Euc})$, even without the ADM mass constraint.

Proof. In this case we use Bochner formula for 1-forms:

$$\Delta\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega).$$

If ω is harmonic, then the left hand side vanishes. To this end, write $x_1, \ldots, x_n$ to be the standard coordinates on $\mathbb{R}^n$, and use the diffeomorphism $\mathbb{R}^n - B_1(0) \cong M - K$ to get the asymptotic coordinates on M, which will also be denoted as $x_1, \ldots, x_n$. Let η be a cutoff function, and then $\eta x_i \in C^\infty(M)$. One can solve the Laplace's equation

$$\begin{cases} \Delta\psi_i = \Delta(\eta x_i), \\ \psi \to 0 \text{ at } \infty. \end{cases}$$

Set $y_i = \eta x_i - \psi_i$. Then $y_i \in C^\infty(M)$ is harmonic and $y_i \to x_i$ at infinity. Moreover, $y_1, \ldots, y_n$ are coordinate functions near infinity.

Let $\omega_i = dy_i$. Then ω_i is harmonic since Δ commutes with d. Hence $\nabla^*\nabla\omega_i + \mathrm{Ric}(\omega_i) = 0$. To use the divergence theorem, we must find a vector field X such that

$$\langle \nabla^*\nabla\omega, \omega \rangle = \langle \nabla\omega, \nabla\omega \rangle + \mathrm{div}(X).$$

Fix $p \in M$ and let $e_1, \ldots, e_n$ be an orthonormal frame near p such that $\nabla_{e_i}(p) = 0$. Then at p,

$$\langle \nabla^*\nabla\omega, \omega \rangle = -\langle \nabla_{e_i}\nabla_{e_i}\omega, \omega \rangle = -e_i \langle \nabla_{e_i}\omega, \omega \rangle + \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega \rangle.$$

Hence, let X be the vector field such that $\langle X, Y \rangle = -\langle \nabla_Y\omega, \omega \rangle$. Then X is the desired vector field. For $\omega = \omega_i$, denote the corresponding $X = X_i$.

Now integration by parts gives

$$\int_{|x|\le R} \langle \nabla^*\nabla\omega_i, \omega_i\rangle = \int_{|x|\le R} \langle \nabla\omega, \nabla\omega\rangle + \int_{|x|=R} \langle X_i, \nu\rangle = \int_{|x|\le R} \langle \nabla\omega, \nabla\omega\rangle - \int_{|x|=R} \langle \nabla_\nu\omega_i, \omega_i\rangle .$$

Hence, by Bochner formula and summation over i,

$$\sum_i \int_{|x|=R} \langle \nabla_\nu\omega_i, \omega_i\rangle = \sum_i \int_{|x|\le R} (|\nabla\omega_i|^2 + \mathrm{Ric}(\omega_i)).$$

Claim: $m_{ADM} = \frac{1}{4\omega_{n-1}} \lim_{R\to\infty} \sum_i \int_{|x|=R} \langle \nabla_\nu\omega_i, \omega_i\rangle$.

Hence, $m_{ADM} = \sum_i \int_M |(\nabla\omega_i|^2 + \mathrm{Ric}(\omega_i)) \ge 0$. The rigidity part is proved as follows. If $m_{ADM} = 0$, then ω_i's are parallel. Since they are orthonormal at infinity, they are orthonormal everywhere, which implies that $\mathrm{Ric} \equiv 0$. Consider the map $M \to \mathbb{R}^n$, $p \mapsto (y_1(p), \ldots, y_n(p))$. Since ω_i's are orthonormal, this is a local isometry. Since M is AF, the map has to be an actual isometry.

Proof of the claim: First note that we can ignore any $O(r^{-2\tau-1})$ part of the integrand, since $\tau > \frac{n-2}{2}$. Next, we compute

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2} g^{im}(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}) \quad (= O(r^{-\tau-1})) \\ &= \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) + O(r^{-2\tau-1}).\end{aligned}$$

Hence,

$$\begin{aligned}\langle \nabla_j\omega_i, \omega_i\rangle &= \left\langle -\Gamma^i_{jk} dy^k, dy^i\right\rangle = -\Gamma^i_{jk} g^{ik} \\ &= -\Gamma^i_{ji} + O(r^{-2\tau-1}) = -\frac{1}{2}\partial_j g_{ii} + O(r^{-2\tau-1}).\end{aligned}$$

Write $\nu = \nu^j \partial_j$. Then we have

$$\lim_{R\to\infty} \sum_i \int_{|x|=R} \langle \nabla_\nu\omega_i, \omega_i\rangle = \lim_{R\to\infty} \sum_{i,j} \int_{|x|=R} \langle \nabla_j\omega_i, \omega_i\rangle \nu^j = \lim_{R\to\infty} \sum_{i,j} \int_{|x|=R} -\frac{1}{2}\partial_j g_{ii}\nu^j.$$

To complete the proof we observe that

$$0 = \Delta y_i = g^{jk}\Gamma^i_{jk} = \partial_j g_{ij} - \frac{1}{2}\partial_i g_{jj} + O(r^{-2\tau-1}).$$

Combining this with the definition of ADM mass, we obtain the desired result. □

Now we are ready to prove the positive mass theorem.

Proof. Step 1: Let ψ_0 be a constant spinor with $|\psi_0| = 1$. We can find a harmonic spinor ψ such that

$$\begin{cases} D\psi = 0, \\ \psi = \psi_0 + \xi, \quad \xi \to 0 \text{ at } \infty. \end{cases}$$

In fact we have an estimate $\xi = O(r^{-\tau})$. We may come back to the analysis part later in this class.

Step 2: Apply Lichnerowicz:

$$0 = D^2\psi = \nabla^*\nabla\psi + \frac{S(g)}{4}\psi.$$

Hence, integration by parts gives

$$0 = \int_{r\le R} \left\langle D^2\psi, \psi \right\rangle = \int_{r\le R} \langle \nabla^*\nabla\psi, \psi\rangle + \left\langle \frac{S(g)}{4}\psi, \psi \right\rangle$$
$$= \int_{r\le R} |\nabla\psi|^2 - \int_{r=R} \langle \nabla_\nu\psi, \psi\rangle + \int_{r\le R} \frac{S(g)}{4} \langle \psi, \psi\rangle .$$

(We pretend to work over $\mathbb{R}$ for simplicity.)

Claim: $\lim_{r\to R}\int_{r=R} \langle \nabla_\nu\psi, \psi\rangle = \omega_{n-1} m_{ADM}$.

Given this we have

$$m_{ADM} = \frac{1}{4\omega_{n-1}} \int_M \left(|\nabla\psi|^2 + \frac{S(g)}{4}|\psi|^2 \right) \ge 0.$$

The rigidity part will be proved next time.

Step 3: Proof of the claim. Apply Gram-Schmidt to $\partial_1, \ldots, \partial_n$, we can find an orthonormal basis $e_1, \ldots, e_n$. Since $g_{ij} = \delta_{ij} + h_{ij}$ where $h_{ij} = O(r^{-\tau})$, we can check that

$$e_i = \partial_i - \frac{1}{2} h_{ik}\partial_k + O(r^{-\tau-1}).$$

Denote ω^{ab} to be the connection 1-forms, i.e., $\nabla e_a = \omega^{ab} e_b$. Then we have, by definition of the spinor connection,

$$\nabla_i\psi = \partial_i\psi + \frac{1}{4}\omega^{ab}(\partial_i)c(e_a)c(e_b)\psi = \partial_i\psi - \frac{1}{8}\partial_k g_{ij}[c(e_j), c(e_k)]\psi + O(r^{-2\tau-1}). \qquad (*)$$

Since $\psi = \psi_0 + \xi$ where $\xi = O(r^{-\tau})$, we have

$$\langle \nabla_i\psi, \xi\rangle = O(r^{-2\tau-1}),$$

and thus

$$\langle \nabla_i\psi, \psi\rangle = \langle \nabla_i\psi_0, \psi_0\rangle + \langle \nabla_i\xi, \psi_0\rangle + O(r^{-2\tau-1}).$$

Apply the formula $(*)$ to ψ_0, we have

$$\langle \nabla_i\psi_0, \psi_0\rangle = -\frac{1}{8}\partial_k g_{ij} \langle [c(e_j), c(e_k)]\psi_0, \psi_0\rangle + O(r^{-2\tau-1}).$$

But $|\psi_0| = 1$ implies that

$$\langle c(e_j)\psi_0, \psi_0\rangle = -\langle \psi_0, c(e_j)\psi_0\rangle .$$

From this we have

$$\langle c(e_k)c(e_j)\psi_0, \psi_0\rangle = -\langle c(e_k)\psi_0, c(e_j)\psi_0\rangle ,$$

and thus

$$\langle [c(e_j), c(e_k)]\psi_0, \psi_0\rangle = 0.$$

Finally we have to deal with the term $\langle \nabla_i\xi, \psi_0\rangle$. Consider the operator

$$L_i = \nabla_i + c(e_i)D = \frac{1}{2}\sum_j [c(e_i), c(e_j)]\nabla e_j.$$

Then we have

$$\int_{r=R} \langle \nabla_i\xi, \psi_0\rangle \nu^i = \int_{r=R} \langle L_i\xi, \psi_0\rangle \nu^i - \int_{r=R} \langle c(e_i)D\xi, \psi_0\rangle \nu^i$$
$$= \int_{r=R} \langle \xi, L_i\psi_0\rangle \nu^i - \int_{r=R} \langle c(e_i)D\psi_0, \psi_0\rangle \nu^i.$$

One can check that $\langle \xi, L_i\psi_0\rangle = O(r^{-2\tau-1})$, and

$$\langle c(e_i)D\psi_0, \psi_0\rangle = -\frac{1}{4}\sum_j(\partial_j g_{ij} - \partial_i g_{jj}) + O(r^{-2\tau-1}).$$

(See Lee-Parker, The Yamabe Problem.)

Hence, we conclude that $\lim_{R\to\infty}\int_{r=R}\langle \nabla_\nu\psi, \psi\rangle = \omega_{n-1}m_{ADM}$. $\square$

7.1 Lecture 11

Recall the PMT

Theorem 7.4. *Let (M^n, g) be a manifold with spin which is AF of order $\tau > (n-2)/2$. If $S(g) \geq 0$ then we have that*

$$m_{ADM} \geq 0$$

where equality holds if and only if $(M^n, g) \cong (\mathbb{R}^n, g_{\text{euclid}})$

We already proved the inequality, given nonnegative scalar curvature. So let us proof the equality case

Proof. Recall that we have shown

$$m_{ADM} = \frac{1}{4\omega n-1}\int_M |\nabla\varphi|^2 + \frac{S(g)}{4}|\varphi|^2 dV_g, \tag{7}$$

where φ is a harmonic spinor asymptotic to the constant spinor φ_0 with $|\varphi_0| = 1$.
Observe that (7) gives us that from $m_{ADM} = 0$, we can infer that φ is a parallel spinor.

Claim: (M^n, g) must be Ricci flat.

Indeed, since $\nabla^S\varphi \equiv 0$, we get that

$$R^S(X, Y)\varphi = 0,$$

so that

$$\frac{1}{4}R_{klij}c(e_i)c(e_j)\varphi = 0 \quad \text{for all } j, k.$$

Hence

$$\frac{1}{4}R_{klij}c(e_l)c(e_i)c(e_j)\varphi = 0 \quad \text{for all } k.$$

Then up to a sign, we get that

$$0 = \sum_{l,i,j \text{ all different}} R_{klij}c(e_l)c(e_i)c(e_j)\varphi \pm \frac{1}{4}2R_{ki}c(e_i)\varphi$$

Now note that by the Bianchi identity, we get that

$$\sum_{l,i,j \text{ all different}} R_{klij}c(e_l)c(e_i)c(e_j)\varphi = 0,$$

from which we may infer that

$$\frac{1}{4}2R_{ki}c(e_i)\varphi = c(R_{ki}e_i)\varphi = 0 \quad \text{for all } k,$$

and hence

$$R_{ki}e_i = 0 \quad \text{for all k}$$

Thus

$$R_{ki} = 0.$$

We thus infer that (M^n, g) is indeed Ricci flat. In order to conclude the claim one makes use of the volume comparison together with the fact that M is AF, which implies the claim. $\square$

8 Compactification

Let (M^n, g) be AF and let M_1 denotes its one point compactification. The main theorem of this chapter is the follwoing:

Theorem 8.1. *Let (M^n, g) be AF of order $\tau > (n-2)/2$ and M_1 its one point compactification. Then if $M_1 \# T^n$ does not have PSC then the PMT holds for (M^n, g), that is, the follwoing implication holds true*

$$S(g) \implies m_{ADM} \geq 0 \quad \text{with } '=' \text{ if and only if } (M, g) \cong (R^n, g_{\text{euclid}}).$$

Corollary 8.2. *For $3 \leq n \leq 7$ one has that the PMT holds.*

Idea of the proof for the above theorem: contraposition.
Assume that that the PMT does not hold, that is,

$$S(g) \geq 0 \quad \text{but} \quad m_{ADM} < 0.$$

Show that $M_1 \# T^n$ will have PSC. To get a PSC metric on $M_1 \# T^n$ we deform g so that it still has nonnegative scalarcurvature and equals the euclidean metric at ∞. We use a conformal deformation

$$\tilde{g} = \psi^{\frac{4}{n-2}} g, \quad \psi > 0, \, \psi \in C^\infty(M)$$

Then we have that

$$S(\tilde{g}) = C_n \psi^{-\frac{n+2}{n-2}} L\psi,$$

where L denotes the conformal Laplacian $L = \Delta + \frac{n-2}{4(n-1)} S(g)$. We will divide the proof into two steps.
Step 1. Solve the equation

$$\begin{cases} L\psi & = 0 \\ \psi & \to 1 \quad \text{at } \infty. \end{cases}$$

This gives that

$$S(\tilde{g}) = 0.$$

To do so, we have the following

Proposition 8.3. *Suppose that (M^n, g) is AF. Then there exists a constant $\varepsilon_0 = \varepsilon_0(g)$ such that for any $f \in C^\infty(M) \cap L^q(M) \cap L^{\frac{2n}{n-2}}(M)$ with $q > \frac{n}{2}$ and $\|f_-\|_{\frac{n}{2}} < \varepsilon_0$. Then the equation*

$$\begin{cases} \Delta u + fu &= 0 \\ u &\to 1 \quad \text{at } \infty \end{cases}$$

has a unique positive solution.

Morever

$$u = 1 + \frac{A}{r^{n-2}} + O(r^{-n+1}) \quad \text{as } r \to \infty,$$

where

$$A = \int_M S(g) f.$$

Remark. During lecture, we imposed the additional condition that $f \in L^1(M)$ to make sense of A.

Proof. Let us consider the problem

$$\begin{cases} \Delta v + fv &= -f \quad \text{on } \Omega_R \\ v|_{\partial\Omega_R} &= 0, \end{cases}$$

where $\Omega_R = \{r \leq R\}$. The idea is to show that there is a solution for each $R > 0$. This follows from the Fredholm alternative. That is, we have to show that the problem

$$\begin{cases} \Delta v + fv &= 0 \quad \text{on } \Omega_R \\ v|_{\partial\Omega_R} &= 0, \end{cases}$$

has only the trivial solution. Indeed, if v is a solution of this problem. Then we get that

$$0 = \int_{\Omega_R} v\Delta v + fv^2 = \int_{\Omega_R} |\nabla v|^2 f v^2$$

This gives that

$$\begin{aligned} \int_{\Omega_R} |\nabla v|^2 &= -\int_{\Omega_R} fv^2 \\ &\leq -\int_{\Omega_R} f_- v^2 \\ &\leq \Big(\int_{\Omega_R} f_-^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_{\Omega_R} v^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}, \end{aligned}$$

where the last inequality follows from the Hölder inequality with

$$\frac{2}{n} + \frac{n-2}{n} = 1.$$

Since we have that $v = 0$ on $\partial\Omega$, we have that in view of the Sobolev inequality that

$$\|v\|^2_{\frac{2n}{n-2}} \leq C_S \|\nabla v\|_2^2$$

Choosing $\varepsilon_0 = \frac{1}{2C_S}$, we get that

$$\nabla v \equiv 0,$$

so that by the boundary condition, we have that

$$v \equiv 0.$$

We thus have that by the Fredholm alternative that

$$\begin{cases} \Delta v + fv & = -f \quad \text{on } \Omega_R \\ v|_{\partial \Omega_R} & = 0, \end{cases}$$

has a unique solution, let us denote it by v_R. The idea is now to pass to the limit as $R \to \infty$. To do so, we want to make use of Arzela Ascoli Theorem. Observe that when integrating over Ω_R, we get that

$$\begin{aligned} \int |\nabla v_R|^2 &= -\int f(v_R^2 + v_R) \\ &\leq -\int f_- v_R^2 - \int f v_R \\ &\leq C_S \|f_-\|_{\frac{n}{2}} \int |\nabla v_R|^2 + \|f\|_{\frac{2n}{n-2}} \|\nabla v_R\|_2 \\ &\leq \frac{1}{2} \|\nabla v_R\|_2^2 + \|f\|_{\frac{2n}{n-2}} \|\nabla v_R\|_2 \end{aligned}$$

From which we get that

$$\int |\nabla v_R|^2 \leq 2\|f\|_{\frac{2n}{n-2}} \|\nabla v_R\|_2$$

and hence

$$\|\nabla v_R\|_2 \leq 2\|f\|_{\frac{2n}{n-2}},$$

where the bound on the RHS is independent of R(!). □

On top of that, we need the following

Lemma 8.4. *Let (M^n, g) be AF. Assume that $u \in C^\infty(M)$, such that $u > 0$ and*

$$u = 1 + \frac{A}{r^{n-2}} + O(r^{-n+1})$$

Then for the metric

$$\tilde{g} = u^{\frac{n}{n-2}} g,$$

one has that

$$m_{ADM}(\tilde{g}) = m_{ADM}(g) + (n-1)A.$$

Now we are in the position to continue the proof of the main theorem. In view of Proposition 8.3, we have that the solution of the equation

$$\begin{cases} Lu & = 0 \quad \text{on } M \\ u & \to 1 \quad \text{at } \infty \end{cases}$$

satisfies

$$u = 1 - \frac{A}{r^{n-2}} + O(r^{-n+1})$$

with

$$A = \int_M S(g)u \geq 0.$$

By definition of conformal Laplacian, we get that

$$g' = u^{\frac{4}{n-1}} g$$

has zero scalarcurvature, being

$$S(g') \equiv 0$$

and is still AF. Moreover, we get that

$$m_{ADM}(g') \leq m_{ADM}(g).$$

We write the metric $g' = \delta_{ij} + a_{ij}$, where $a_{ij} = O(r^{-\tau})$. Then, let us introduce a cutoff function $\rho : [0, \infty) \to \mathbb{R}$ such that

$$\rho \equiv 1 \quad \text{on } [0, R] \quad \text{and} \quad \rho \equiv 0 \quad \text{on } [2R, \infty).$$

Then we define the metric $\hat{g}$ to be given by

$$\hat{g} = \begin{cases} g_{euclid} & \text{for } r \geq 2R \\ g' & \text{for } r \leq R. \end{cases}$$

Then from

$$S(\hat{g}) = O(R^{-\tau}),$$

we can choose R large enough such that

$$\|S(\hat{g})\|_{\frac{n}{2}} < \varepsilon_0.$$

We therefore have that the conditions for Proposition 8.3 are satisfied applied to the equation

$$\begin{cases} L_{\hat{g}} v & = 0 \quad \text{in } M \\ v & \to 1 \quad \text{at } \infty. \end{cases}$$

and put $\tilde{g} = v^{\frac{4}{n-2}} \hat{g}$, which will have not scalarcurvature, being

$$S(\tilde{g})] \equiv 0.$$

Moreover

$$\tilde{g_{ij}} = v^{\frac{4}{n-2}} \delta_{ij} \quad \text{for } r \geq 2R.$$

By our assumption, $m_{ADM}(g) < 0$, we wonder wether this still holds for $m_{ADM}(\tilde{g})$. Let us make a claim.
Claim: We get that

$$m(\tilde{g}) \to m(g') \quad \text{as } R \to \infty,$$

where we just wright $m(h) = m_{ADM}(h)$ for any metric on M.
Form this we geth that

$$m(\tilde{g}) < 0 \quad \text{for sufficiently large } R.$$

Recall that our solution v satisfies

$$v = 1 + \frac{A}{r^{n-2}} + O(r^{-n+1})$$

While from Lemma **??**, we get that

$$0 < m(\tilde{g}) = m(g_{euclid}) + A(n-1),$$

from which we get that

$$A < 0.$$

The next step is to modify the metric one more time. Indeed, let us find R large enough and $\varepsilon_0 > 0$ small enough such that

$$\sup_{r=R} v \leq 1 - \varepsilon_0 \quad \text{and} \qquad \inf_{r \geq 2R} v \geq 1 - \frac{\varepsilon_0}{2}.$$

Then we consider the super harmonic function

$$w := \min\{v, 1 - \tfrac{\varepsilon_0}{2}\}$$

The idea is now to smooth it out in such a way that it stays super harmonic, still denoted by w. Then define the metric

$$\overline{g} = \begin{cases} \tilde{g} & r \leq R \\ w^{\frac{4}{n-2}} \delta_{ij} & r > 2R. \end{cases}$$

To summarize the development so far: we begin with an AF Riemannian manifold (M, g) with $S(g) \geq 0$ and $m(g) < 0$. We first deform (M, g) to $(M, \overline{g})$ with $S(\overline{g}) \geq 0$ and $\overline{g} = g_{\text{Euclid}}$ outside a ball of radius $2R$ for some R. Then we compactify and take the connected sum $M_1 \# T^n$ to obtain a PSC on $M_1 \# T^n$. (Figure forthcoming.) We achieve this with the following sequence of metric changes:

$$g \longrightarrow g' = u^{\frac{4}{n-2}} g \longrightarrow \hat{g} \longrightarrow \tilde{g} = v^{\frac{4}{n-2}} \hat{g}.$$

The transition from g' to $\hat{g}$ is accomplished via a cutoff function ρ with $\rho \equiv 1$ on $[0, R]$ and $\rho \equiv 0$ on $[2R, \infty)$ where R is such that $m(\tilde{g}) < 0$. Then we will have $S(\tilde{g}) = 0$ and $m(\tilde{g}) < 0$. Because $m(\tilde{g}) < 0$ we can introduce a new function w by smoothing out $\min\{v, 1 - \frac{\epsilon_0}{2}\}$ in such a manner that $w \equiv v$ for $r \leq R$ and $w \equiv 1 - \frac{\epsilon_0}{2}$ for $r \geq 2R$, and that w stays superharmonic, $\triangle w \geq 0$. Then define

$$\overline{g} = w^{\frac{4}{n-2}} \hat{g};$$

this will be the final $(M, \overline{g})$ we wanted before compactification.

The new metric $\overline{g}$ has scalar curvature

$$S(\overline{g}) = c_n w^{-\frac{n+2}{n-2}} \left(\triangle_{\hat{g}} w + S(\hat{g}) w \right) = c_n w^{-\frac{n+2}{n-2}} \triangle_{\hat{g}} w \geq 0$$

since $w \geq 0$ and w is superharmonic. Also v is nonconstant so we get $S(\overline{g}) > 0$ somewhere on $r \leq R$. Hence by compactifying we can pass to $M_1 \# T^n$ with a metric, still denoted $\overline{g}$, such that $S(\overline{g}) \geq 0$

and $S(\overline{g}) > 0$ somewhere. Indeed, this is good enough to get PSC on $M_1 \# T^n$: the first eigenvalue λ_1 of

$$L = \triangle_{\overline{g}} + \frac{n-2}{4(n-1)} S(\overline{g})$$

is positive, $\lambda_1 > 0$, and we can take the first eigenfunction φ of L to be positive. So if

$$g'' := \varphi^{\frac{4}{n-2}} \overline{g}$$

then

$$S(g'') = c_n \lambda_1 \varphi^{-\frac{4}{n-2}} > 0,$$

which shows that the metric g'' on $M_1 \# T^n$ has PSC.

So far we have shown that if $M_1 \# T^n$ has no PSC metric, then any metric g on M with nonnegative scalar curvature must have nonnegative ADM mass. To finish proving that PMT holds for M, we still need the rigidity result: if $m(g) = 0$, then (M, g) is isometric to $(\mathbb{R}^n, g_{\text{Euclid}})$.

Suppose that g is a metric on M with $m(g) = 0$. We assert that $S(g) \equiv 0$. Suppose on the contrary that $S(g)$ does not uniformly vanish. Solve

$$\begin{cases} \triangle_g u + \frac{n-2}{4(n-1)} S(g) u = 0 \\ u \to 1 \text{ at } \infty \end{cases} \qquad \Longrightarrow u = 1 + \frac{A}{r^{n-2}} + O(r^{-n+1}).$$

Then $\tilde{g} := u^{\frac{4}{n-2}} g$ has scalar curvature $S(\tilde{g}) \equiv 0$ and mass

$$m(\tilde{g}) = m(g) + (n-1)A = -(n-1)c_n \int_M S(g) u < 0. \tag{8}$$

But this contradicts the PMT.

Now we show that in fact $\operatorname{Ric}(g) \equiv 0$. Let h be a compactly supported symmetric 2-tensor and set $g_t := g + th$. For each t, solve

$$\begin{cases} \triangle_{g_t} u_t + \frac{n-2}{4(n-1)} u_t = 0 \\ u_t \to 1 \text{ at } \infty \end{cases},$$

and let $\tilde{g}_t = u_t^{\frac{4}{n-2}}$; these metrics each have $S(\tilde{g}_t) \equiv 0$. And, each metric has mass, denoted $m(t)$, given by

$$m(t) = m(\tilde{g}_t) = (n-1)A_t, \ A_t := -c_n \int_M S(g_t) u_t.$$

We claim that

$$m'(0) = c_1(n) \int_M \langle \operatorname{Ric}(g), h \rangle.$$

To prove the claim we refer to the following formula:

$$\frac{d}{dt} S(g_t) = \triangle_{g_t} \left(\operatorname{tr}_{g_t}(\dot{g}_t) \right) + \operatorname{div}_{g_t} \left(\operatorname{div}_{g_t} \dot{g}_t \right) - \langle \operatorname{Ric}(g_t), \dot{g}_t \rangle .$$

Remark. Let $A,\ B$ be 2-tensors, with components $A_{ij},\ B_{ij}$ with respect to a local frame $\{E_i\}$. Then $\langle A, B \rangle := g^{ik} g^{jl} A_{ij} B_{kl}$.

9 A Closer Look at Scalar Curvature

Recall that for the different notions of curvature,

$$\text{Sectional} \;\geq\; \text{Ricci} \;\geq \text{Scalar};$$

equality holds for $n = 2$ where they are all the same. For $n \geq 3$, "manifolds prefer negative Ricci or scalar curvature."

Theorem 9.1 (Aubin)**.** *Every closed manifold M^n, $n \geq 3$, has a metric of constant negative scalar curvature.*

Theorem 9.2 (Lohkamp)**.** *Every manifold M^n, $n \geq 3$ has a complete metric of negative Ricci curvature.*

So negative Ricci or scalar curvature does not impose any topological restriction. Of course we know the same is not true for positive (and bounded above 0) Ricci curvature by the Bonnet-Myers theorem. For scalar curvature we have:

Question: What manifolds have PSC?

Theorem 9.3 (Licnherowicz)**.** *M^n closed, spin, and PSC $\Rightarrow \hat{A}(M) = 0$.*

Here $\hat{A}(M) =$"roof genus" is a topological invariant of M.

For simply connected manifolds with $n \geq 5$ the question is completely answered.

Theorem 9.4 (Gromov-Lawson, Stolz)**.** *M^n simply connected and $n \geq 5$. Then M has PSC metric if and only if either M is not spin, or M is spin and the α-invariant vanishes.*

Remark. The α-invariant is related to the index of the Dirac operator. In fact if $n \equiv 0 \mod 4$ then the α-invariant is the same as $\hat{A}(M)$.

Remark. There is an exotic S^9 with $\alpha \neq 0$; this exotic S^9 has no PSC,

Only 2 cases remain: low dimension ($n = 3,\ 4$) and not simply connected (e.g., T^n). For $n = 3$ there is a complete solution (thanks to Perelman's solution of Poincaré conjecture and Agol's solution of Virtual Haken conjecture).

Theorem 9.5. *A closed orientable 3-manifold has PSC if and only if it is the connected sum of the spherical space form and $S^2 \times S^1$.*

Remark. The spherical space forms are S^3/Γ for Γ a free action on S^3, e.g. the Lens spaces.

The case $n = 4$ is quite open.

10 Enlargeability

Last time: Let (M, g) AF with nonnegative scalar curvature. Then $M_1 \# T^n$ has no PSC implies that the PMT holds on M.

We proved earlier in the term that if $3 \leq n \leq 7$ and M^n is closed then $M^n \# T^n$ has no PSC. So for $3 \leq n \leq 7$ we get PMT without needing M to be spin. Our current goal is to show that if M is closed and spin then $M^n \# T^n$ has no PSC, which gives a proof of PMT for the spin case.

Definition 10.1. Suppose $f : X^n \to Y^n$ is C^1 and $\epsilon > 0$. We say f is *ϵ-contractible* if for all $p \in X$, $f_* : T_pY \to T_{f(p)}Y$ is ϵ-contractible; that is, for all $v \in T_pX$, $\|f_* v\|_Y \leq \epsilon \|v\|_X$.

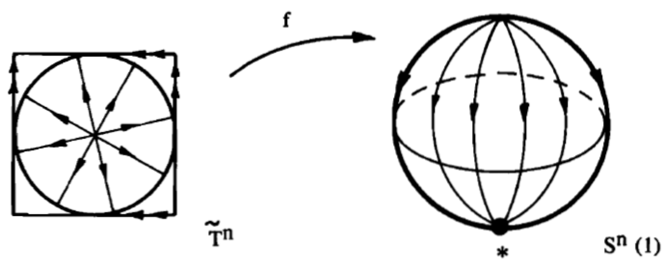
Example 10.2. For any $\epsilon > 0$ there is an ϵ-contractible map $f; \mathbb{R}^n \to S^n(1)$.

Definition 10.3. A compact riemannian n-manifold is said to be enlargeable if for every $\varepsilon > 0$ there exists an orientable riemannian covering space which admits an ε-contracting map onto $S^n(1)$ which is constant at infinity and of non-zero degree. If for each $\varepsilon > 0$, there is a finite covering space with these properties, we call the manifold compactly enlargeable.

Remark. A map is constant at infinity if it is constant outside a compact set. The degree of such a map $f : X \to S^n$ is defined as

$$\deg(f) = \frac{\int_X f^*\omega}{\int_{S^n} \omega}$$

where ω is an n-form on S^n with non-zero integral. The degree can also be defined as usual in terms of signed counting of pre-images of of f at regular values.



The square flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is certainly enlargeable since the universal covering space has the required mappings for all $\varepsilon > 0$. This torus is, in fact, compactly enlargeable. We see this as follows. For each $k \in \mathbb{Z}^+$, the lattice $(k \cdot \mathbb{Z})^n \subset \mathbb{Z}^n$ gives a k^n-fold covering torus $\tilde{T}^n \equiv \mathbb{R}^n/(k \cdot \mathbb{Z})^n$, which admits the (π/k)-contracting map to $S^n(1)$ of degree 1 pictured above.

Theorem 10.1. *The following statements hold in the category of compact manifolds:*

(A) Enlargeability is independent of the riemannian metric.

(B) Enlargeability depends only on the homotopy-type of the manifold.

(C) The product of enlargeable manifolds is enlargeable.

(D) The connected sum of any manifold with an enlargeable manifold is again enlargeable.

(E) Any manifold which admits a map of non-zero degree onto an enlargeable manifold is itself enlargeable.

Proof. It is evident that $(E) \Rightarrow (B) \Rightarrow (A)$ and that $(E) \Rightarrow (D)$. To prove (E) we consider two compact oriented riemannian n-manifolds X and Y, and a map $F : X \to Y$ of non-zero degree. By compactness there exists a $c > 0$ so that $\|dF\| \leqq c$ on X (i.e., F is c-contracting). Given $\varepsilon > 0$, there is a riemannian covering space $p : \tilde{Y} \to Y$ which admits a (ε/c)-contracting map $f : \tilde{Y} \to S^n(1)$ which is constant outside a compact set $\tilde{K} \subset \tilde{Y}$ and of nonzero degree. Taking the fibre product of p and F gives a covering space $p' : \tilde{X} \to X$ and a proper mapping $\tilde{F} : \tilde{X} \to \tilde{Y}$ so that the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{F}"] \arrow[d, "p'"] & \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

commutes. Since $\tilde{F}$ is a lifting of F, we have $\|\nabla\tilde{F}\| \leqq c$ on $\tilde{X}$. Hence, the composition $f \circ \tilde{F} : \tilde{X} \to S^n(1)$ is ε-contracting. Since $\tilde{F}$ is proper, we see that $f \circ \tilde{F}$ is constant outside the compact set $\tilde{F}^{-1}(\tilde{K})$. It is easy to see that: $\deg(f \circ \tilde{F}) = \deg(f)\deg(F) \neq 0$. Hence, X is enlargeable as claimed.

To prove (C), we fix a degree-1 map $\phi : S^n(1) \times S^m(1) \to S^{n+m}(1)$ (Recall that $S^{n+m} \cong S^n \times S^m / S^m \vee S^n$) and let $c = \sup \|d\phi\|$. This map is chosen to be constant on the set $(S^n(1) \times \{*\}) \cup (\{*\} \times S^m(1))$, where each "*" denotes a distinguished point in the sphere. Suppose now that we are given (ε/c)-contracting maps, $f : X^n \to S^n(1)$ and $g : X^m \to S^m(1)$, which are constant $(= *)$ at infinity and of non-zero degree. Then the map $\phi \circ (f \times g) : X^n \times Y^m \longrightarrow S^{n+m}(1)$ is ε-contracting, constant at infinity and of non-zero degree. From here the argument is straightforward. □

Theorem 10.2. *An enlargeable spin manifold X cannot carry a metric of positive scalar curvature.*

10.1 Review on index Theorem and Lichnerowicz Formula

Theorem 10.3. *Let M be a closed Spin manifold, $S \to M$ be the spinor bundle with spinor connection ∇^S. Let $E \to M$ be a complex vector bundle with a unitary connection ∇^E. On $S \otimes E$, one has connection $\nabla^{S\otimes E} := \nabla^S \otimes 1 + 1 \otimes \nabla^E$, i.e., for any $s \in \Gamma(s), e \in \Gamma(E)$, $\nabla^{S\otimes E} s \otimes e = \nabla^S s \otimes e + s \otimes \nabla^E e$. Also, $S \otimes E$ admits a clifford acction, such that for any $X \in \Gamma(TM)$, $c(X) s \otimes e = (c(X)s) \otimes e$. Let $D^{S\otimes E} := \sum_i c(e_i)\nabla^{S\otimes E}_{e_i}$ be the Dirac operator (where $\{e_i\}$ is a local orthonormal frame), then*

$$ind(D^{S\otimes E}) = \int_M \hat{A}(M) ch(E).$$

Theorem 10.4 (Lichnerowicz formula)**.**

$$(D^{S\otimes E})^2 = \Delta + k/4 + \mathcal{R}^E,$$

where k is the scalar curvature of M, Δ is the connection Laplacian with respect to $\nabla^{S\otimes E}$, $\mathcal{R}^E := \sum_{i,j} c(e_i)c(e_j)R^E(e_i, e_j)$, R^E is the curvature on E.

10.2 Quick introduction to Chern-Weil theory

Let $E \to M$ be a smooth complex vector bundle over a smooth compact manifold M. We denote by $\Omega^*(M; E)$ the space of smooth sections of the tensor product vector bundle $\Lambda^*(T^*M) \otimes E$ obtained from $\Lambda^*(T^*M)$ and E :

$$\Omega^*(M; E) := \Gamma(\Lambda^*(T^*M) \otimes E).$$

Definition 10.4. A connection ∇^E on E is a $\mathbb{C}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M; E)$ such that for any $f \in C^\infty(M), X \in \Gamma(E)$, the following Leibniz rule holds,

$$\nabla^E(fX) = (df)X + f\nabla^E X.$$

Just like the exterior differential operator d, a connection ∇^E can be extended canonically to a map, which we still denote by ∇^E,

$$\nabla^E : \Omega^*(M; E) \longrightarrow \Omega^{*+1}(M; E)$$

such that for any $\omega \in \Omega^*(M), X \in \Gamma(E)$,

$$\nabla^E : \omega X \mapsto (d\omega)X + (-1)^{\deg \omega}\omega \wedge \nabla^E X$$

Definition 10.5. The curvature R^E of a connection ∇^E is defined by

$$R^E = \nabla^E \circ \nabla^E : \Gamma(E) \to \Omega^2(M; E),$$

which, for brevity, we will write $R^E = \left(\nabla^E\right)^2$.

One can see that R^E may be thought of as an element of $\Gamma(\mathrm{End}(E))$ with coefficients in $\Omega^2(M)$. In other words,

$$R^E \in \Omega^2(M; \mathrm{End}(E))$$

To give a more precise formula, if $X, Y \in \Gamma(TM)$ are two smooth sections of TM, then $R^E(X,Y)$ is an element in $\Gamma(\mathrm{End}(E))$ given by

$$R^E(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}.$$

Finally, in view of the composition of the endomorphisms, one sees that for any integer $k \geq 0$,

$$\left(R^E\right)^k = \overbrace{R^E \cdot \cdots \cdot R^E}^{k} : \Gamma(E) \longrightarrow \Omega^{2k}(M; E)$$

is a well-defined element lying in $\Omega^{2k}(M; \mathrm{End}(E))$.

For any smooth section A of the bundle of endomorphisms, End (E), the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map

$$\mathrm{tr} : \Omega^*(M; \mathrm{End}(E)) \longrightarrow \Omega^*(M)$$

such that for any $\omega \in \Omega^*(M)$ and $A \in \Gamma(\mathrm{End}(E))$,

$$\mathrm{tr} : \omega A \mapsto \omega\, \mathrm{tr}[A].$$

We still call it the function of trace.

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

be a power series in one variable. Let R^E be the curvature of a connection ∇^E on E. The trace of

$$f\left(R^E\right) = a_0 + a_1 R^E + \cdots + a_n \left(R^E\right)^n + \cdots$$

is an element in $\Omega^*(M)$. We can now state a form of the Chern-Weil theorem as follows.

Theorem 10.5. *(i) The form* $\mathrm{tr}\left[f\left(R^E\right)\right]$ *is closed. That is,*

$$d\,\mathrm{tr}\left[f\left(R^E\right)\right] = 0.$$

(ii) If $\widetilde{\nabla}^E$ *is another connection on* E *and* $\widetilde{R}^E$ *its curvature, then there is a differential form* $\omega \in \Omega^*(M)$ *such that*

$$\mathrm{tr}\left[f\left(R^E\right)\right] - \mathrm{tr}\left[f\left(\widetilde{R}^E\right)\right] = d\omega.$$

Since

$$\det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right) = \exp\left(\mathrm{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right]\right)$$

in view of the following power series expansion formulas for $\log(1+x)$ and $\exp(x)$

$$\log(1+x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

and

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots .$$

By Theorem 10.5, the Chern class

$$c(E) = \left[\det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in H^*(M, \mathbb{C})$$

is some summation of even cohomologies, i.e., one has

$$c(E) = 1 + c_1(E) + \cdots + c_k(E) + \cdots$$

with each i-th Chern class

$$c_i(E) \in H^{2i}(M).$$

(Here for a closed differential form w, $[w]$ denotes the cohomology represented by w.)

Similarly, the Chern character and $\hat{A}$-class are defined by

$$\mathrm{ch}(E,) = \left[\mathrm{tr}\left(\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right)\right] \in H^{\mathrm{even}}(M),$$

$$\widehat{A}(E) = \left[\det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right)^{1/2}\right)\right].$$

It follows from the definition that $\hat{A}_0(E) = 1$.

Moreover, $\hat{A}(M) := \hat{A}(TM \otimes \mathbb{C})$.

Definition 10.6. For $[w] \in H^*(M)$, we define the pairing

$$\langle [w], [M] \rangle := (\int_M w =) \int_M w_n,$$

where w_n is the top degree components of w.

10.3 Proof of Theorem 10.2

For clarity's sake we only present here a proof for the case of compactly enlargeable manifolds.

Let X be a compactly enlargeable n-manifold, and suppose X carries a metric with $\kappa \geqq \kappa_0$ for a constant $\kappa_0 > 0$. We may assume that X has even dimension $2n$. (If not, replace X by $X \times S^1$.)

Choose a complex vector bundle E_0 over the sphere $S^{2n}(1)$ with the property that the top Chern class $c_n(E_0) \neq 0$. (This is certainly possible, cf. [1]). We now fix a unitary connection ∇^{E_0} on E_0 and we let R^{E_0} denote the curvature 2-form. Moreover,

$$\mathrm{ch}(E_0) = rank(E_0) + \frac{1}{(n-1)!}c_n(E_0).$$

Let $\varepsilon > 0$ be given and choose a finite orientable covering $\tilde{X} \to X$ which admits an ε-contracting map $f : \tilde{X} \to S^{2n}(1)$ of non-zero degree. Using f, we pull back the bundle E_0, with its connection, to $\tilde{X}$. This gives us a bundle $E \equiv f^*E_0$ with connection $\nabla^E \equiv f^*\nabla^{E_0}$. We then consider the complex spinor bundle S of $\tilde{X}$ with its canonical riemannian connection, and consider the Dirac operator $D^{S\otimes E}$ on the tensor product $S \otimes E$. We know from Theorem 10.4 that

$$(D^{S\otimes E})^2 = \Delta + \frac{k}{4} + \mathcal{R}^E$$

where $\mathcal{R}^E$ depends universally and linearly on the components of the curvature tensor R^E of E, Moreover

$$\left\|\mathcal{R}^E\right\| \leq C|f_*|^2|R^{E_0}| \leq C'\varepsilon^2$$

for some $C > 0$.

Hence if ε is small, by Theorem 10.4, $D^{S\otimes E}$ is invertible, hence $ind(D^{S\otimes E}) = 0$.

However, let $m = rank(E_0)$ this index is given by

$$
\begin{aligned}
\operatorname{ind}\left(D^{S\otimes E}\right) &= \langle \operatorname{ch} E \cdot \hat{A}(\tilde{X}), [\tilde{X}]\rangle \\
&= \int_{\tilde{X}} \left(m + \frac{1}{(n-1)!} c_n(E)\right) \cdot \hat{A}(\tilde{X}) \\
&= m \int_{\tilde{X}} \hat{A}_n(\tilde{X}) + \int_{\tilde{X}} \frac{1}{(n-1)!} c_n(E) \hat{A}_0 \\
&= \int_{\tilde{X}} \frac{1}{(n-1)!} c_n\left(f^* E_0\right) \text{ (By Theorem 6.13)} \\
&= \int_{\tilde{X}} \frac{1}{(n-1)!} f^*\left(c_n\left(E_0\right)\right) \\
&= \frac{1}{(n-1)!} \deg(f) \int_{S^{2n}} c_n\left(E_0\right) \\
&\neq 0,
\end{aligned}
$$

which is a contradiction.

10.4 Positive Scalar Curvature and Enlargeability

Recall that in the proof of Theorem 10.2, the story of the index of the Dirac operator contains two parts:

1. **Geometric part:** Positive scalar curvature + enlargeability

   $\Longrightarrow$ By the BLW (Bochner - Lichnerowicz - Weitzenböck) formula: $D_E^2 = \nabla^*\nabla + \frac{S}{4} + \mathfrak{R}^E$, we have $\ker D_E = 0$

   $\Longrightarrow \operatorname{ind} D_E^+ = 0$.

2. **Topological part:** Index formula $\Longrightarrow \operatorname{ind} D_E^+ \neq 0$.

And we have seen that this argument works very well on compactly enlargeable manifolds, i.e. the covering space in the definition of enlargeability can be chosen to be finite-sheeted.

Question: Consider M^n to be a compact Riemannian manifold with $K \leq 0$. Can M have a metric with positive scalar curvatire?

Answer: Before we give an answer to this question, let us first examine some examples.

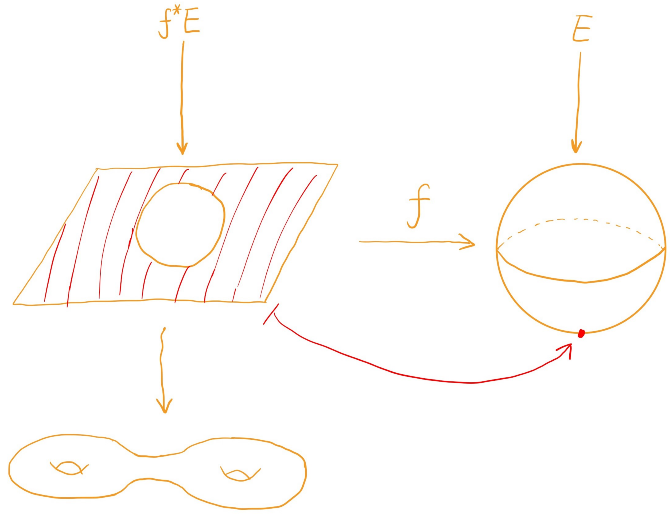
- The n-dimensional torus T^n with $K = 0$: One can choose a finite cover. So our argument above works for this situation.
- Any hyperbolic manifold M: Its universal cover is $\tilde{M} \cong \mathbb{R}^n$, which is noncompact. And as we will see later, it is precisely this noncompactness that brings us the difficulties.

1. **Index ?** On compact manifolds, elliptic operators, such as Laplacian and Dirac operators, give us Fredholm operators. And thus the index of such operators can be defined to be

$$\operatorname{ind} P := \dim\ker P - \dim\operatorname{coker} P.$$

But we are no longer that lucky when it comes to noncompact manifolds, since the dimensions involved might be infinite. However, in the situation that M is a closed spin enlargeable manifolds, then we can overcome this problem as follows:

Since M itself is compact, the scalar curvature on M has a positive lower bound: $S(g) \geq \delta > 0$. This implies after passing to the covering space $\tilde{M}$, one still have a uniform scalar curvature lower bound: $S(\tilde{g}) \geq \delta > 0$.



Now recall that the map $f : \tilde{M} \to S^n$ is constant at infinity, which implies that $\mathfrak{R}^{f^*E} = 0$ outside $B_R(0)$. So by the BLW formula $\tilde{D}^2_{f^*E} = \nabla^*\nabla + \frac{S(\tilde{g})}{4} + \mathfrak{R}^{f^*E}$ one can conclude that $\tilde{D}^2_{f^*E} \geq \frac{\delta}{4} > 0$ outside $B_R(0)$. Therefore the Dirac operator $\tilde{D}_{f^*E}$ is Fredholm and thus its index $\operatorname{ind} \tilde{D}_{f^*E}$ is well-defined.

2. **Index formula ?** Usually the index formula on noncompact manifolds are very complicated, and depends on the geometric property at infinity. But thanks to Gromov-Lawson, we have the so-called Relative Index Theorem that plays the role of the Atiyah-Singer Index Theorem. As suggested by the name of the theorem, the idea here is to consider the difference of two indexes with the same geometric information at infinity.

Consider another map $\tilde{f} : \tilde{M} \to S^n$, where $\tilde{f} \equiv$ const. $= \{$South pole$\}$. Then the pullback bundle $\tilde{f}^*E$ becomes a trivial vector bundle outside some compact region. And from the definition we can easily see that $f = \tilde{f}$ outside $B_R(0)$.

Again notice that in the BLW formula $\tilde{D}^2_{\tilde{f}^*E} = \nabla^*\nabla + \frac{S(\tilde{g})}{4} + \mathfrak{R}^{\tilde{f}^*E}$, the operator $\mathfrak{R}^{\tilde{f}^*E}$ is identically zero outside $B_R(0)$. Therefore we can deduce that $\tilde{D}_{\tilde{f}^*E} \geq \frac{1}{4}\delta > 0$, which implies the Dirac operator $\tilde{D}_{\tilde{f}^*E}$ is Fredholm. Hence $\operatorname{ind} \tilde{D}_{\tilde{f}^*E}$ is well-defined.

Now we have two Dirac operators and two well-defined indexes. It turns out that their difference $\operatorname{ind}(\tilde{D}_{f^*E}) - \operatorname{ind}(\tilde{D}_{\tilde{f}^*E})$ has a pretty nice formula:

Theorem 10.7 (Gromov-Lawson's Relative Index Theorem)**.**

$$\operatorname{ind}(\tilde{D}^+_{f^*E}) - \operatorname{ind}(\tilde{D}^+_{\tilde{f}^*E}) = \int_{\tilde{M}} \hat{A}(T\tilde{M}) \wedge \Big(\operatorname{ch}(f^*E) - \operatorname{ch}(\tilde{f}^*E)\Big).$$

And using this Relative Index Theorem, we can eventually prove

Theorem 10.8 (Gromov-Lawson)**.** *Suppose M^n is a closed manifold with $K \leq 0$, then M cannot have a metric with positive scalar curvature.*

10.5 Positive Mass Theorem for $AF + X$ manifolds

In this section let us consider a more general situation. Let $\bar{M}$ be a manifold constructed by connecting a compact manifold with some noncompact "ends". And we require one of the ends to

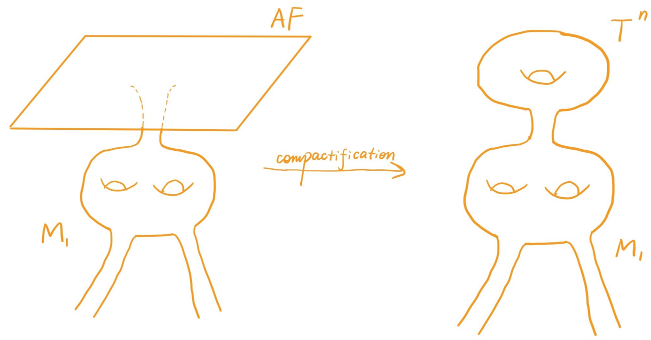


Figure 1: An "alien" whose head is asymptotic flat An "alien" whose head is compact

be asymptotic flat, but without any control for any other ends. As usual, we can still compactify the asymptotic flat end and thus get $M_1 \# T^n$.

Question: Can $M := M_1 \# T^n$ have positive scalar curvature?

Answer: No. So Positive Mass Theorem also holds for $AF + X$-type manifolds.

Difficulty: No uniform scalar curvature lower bound.

So how to make sense of index? First we need to add a perturbation term to the Dirac operator as

$$\tilde{D}_{f^*E} = c(e_i)\nabla^{\slashed{S}}_{e_i} \otimes 1 + c(e_i)\nabla_{e_i} f^*E,$$

where the last term is the perturbation.

Next modification: Add a potential term! But first, we need to define the relative index (sometimes it is also called the super index).

Recall that under an orthonormal frame, the usual Dirac operator can be expressed as

$$D = c(e_i)\nabla^{\slashed{S}}_{e_i} : \Gamma(\slashed{S}) \to \Gamma(\slashed{S}),$$

where $\slashed{S} \to M$ is the spinor bundle, $\{e_i\}$ is a local orthonormal frame on M, and $c(e_i) : \slashed{S} \to \slashed{S}$ is the Clifford multiplication. Since $c(e_i)$ satisfy the relations

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij},$$

one can easily check that for any $X, Y \in \Gamma(TM)$ we have

$$c(X)c(Y) + c(Y)c(X) = -2\langle X, Y\rangle.$$

In particular, we have $c(X)^2 = -|X|^2$.

If the dimension of M is even, then we can consider the operator $\omega := (-1)^{\frac{n(n+1)}{4}} c(e_1)\cdots c(e_n)$. One can check that

- The definition of ω is independent of the choice of the orthonormal basis
- $\omega^2 = 1 \implies$ A $\mathbb{Z}_2$-grading: $\slashed{S} = \slashed{S}^+ \oplus \slashed{S}^-$, where $\slashed{S}^+$ (or $\slashed{S}^-$) is the eigenspace of ω associated with the eigenvalue 1 (or -1 respectively).
- $\omega D = -D\omega \implies D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : \slashed{S}^\pm \to \slashed{S}^\mp$.
- D is self-adjoint $\implies (D^\pm)^* = D^\mp$.

- $\operatorname{ind} D^+ = \dim\ker D^+ - \dim\operatorname{coker} D^+$
$$= \dim\ker D^+ - \dim\ker (D^+)^*$$
$$= \dim\ker D^+ - \dim\ker D^-.$$

- $\ker D = 0 \iff \ker D^\pm = 0 \implies \operatorname{ind} D^+ = 0.$

All the arguments also hold true for $D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}$.

Now suppose E_1, E_2 are two Hermitian vector bundles with unitary connections over M. Let $E = E_1 \oplus E_2$.

$$\begin{aligned} \operatorname{ind} D_{E_1}^+ - \operatorname{ind} D_{E_2}^+ &= \left(\dim\ker D_{E_1}^+ - \dim\ker D_{E_1}^-\right) - \left(\dim\ker D_{E_2}^+ - \dim\ker D_{E_2}^-\right) \\ &= \left(\dim\ker D_{E_1}^+ + \dim\ker D_{E_2}^-\right) - \left(\dim\ker D_{E_1}^- + \dim\ker D_{E_2}^+\right) \end{aligned}$$

Thus we get something that looks like an index!

To make it an actual index of some operator, let

$$D_E^+ := \begin{pmatrix} 0 & D_{E_2}^- \\ D_{E_1}^+ & 0 \end{pmatrix} : \Gamma(\$^+ \otimes E_1) \otimes \Gamma(\$^- \otimes E_2) \to \Gamma(\$^- \otimes E_1) \otimes \Gamma(\$^+ \otimes E_2).$$

Then

$$\ker D_E^+ = \ker D_{E_1}^+ \oplus \ker D_{E_2}^-.$$

Similarly let $(D_E^+)^* := D_E^- = \begin{pmatrix} 0 & D_{E_1}^- \\ D_{E_2}^+ & 0 \end{pmatrix}$. Then Then we get

$$\operatorname{ind} D_E^+ = \operatorname{ind} D_{E_1}^+ - \operatorname{ind} D_{E_2}^+.$$

Consider the map $\sigma : E \to E$ defined by $\sigma|_{E_1} \equiv 1$ and $\sigma|_{E_2} \equiv -1$. Recall that ω can be viewed as a map $\$ \to \$$ satisfying $\omega^2 = 1$. Therefore the map $\omega \otimes \sigma : \$ \otimes E \to \$ \otimes E$ also satisfies $(\omega \otimes \sigma)^2 = 1$. This again induces a $\mathbb{Z}_2$-grading:

$$\$ \otimes E = \Big((\$^+ \otimes E^+) \oplus (\$^- \otimes E^-) \Big) \oplus \Big((\$^+ \otimes E^-) \oplus (\$^- \otimes E^+) \Big),$$

where $(\$^+ \otimes E^+) \oplus (\$^- \otimes E^-)$ and $(\$^+ \otimes E^-) \oplus (\$^- \otimes E^+)$ are the eigenspaces of $\omega \otimes \sigma$ associated with eigenvalues 1 and -1 respectively.

Remark. Characteristic feature of the index of Fredholm operators: homotopy invariance. Let P_t be a continuous family of Fredholm operators. Then we have $\operatorname{ind}(P_t) \equiv \mathit{const}$.

Now we can deform D_E by adding a potential term. Let X be a vector field on S^n, then the Clifford multiplication $c(X) : \$(S^n) \to \$(S^n)$ satisfies

- $c(X)^2 = -|X|^2$
- $c(X) : \$^+(S^n) \to \$^-(S^n)$

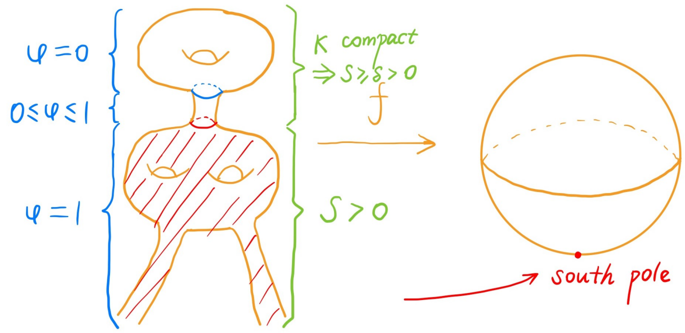
Define $V = \begin{pmatrix} 0 & -c(X) \\ c(X) & 0 \end{pmatrix} : \$(S^n) \to \$(S^n)$. Then V is a self-adjoint operator which satisfies $V^2 = |X|^2$. For any $\varepsilon > 0$, set X such that $X = X_0 \neq 0$ at the south pole, and a cut-off function φ. Now we are able to define the deformation:

$$D_{E,V} = D_E + \varepsilon\varphi \cdot f^* V.$$

It follows directly from the definition that

$$(D_{E,V}^2 = D_E^2 + \epsilon^2 |X_0|^2 \geq \epsilon^2 |X_0|^2 > 0, \text{ outside } K.$$

So finally we get a Fredholm operator! Therefore once again the index $\operatorname{ind} D_{E,V}^+$ is well-defined. We have the following beautiful formula concerning the index of this new Dirac operator



Theorem 10.9 (W. Zhang)**.**

$$\operatorname{ind} D^+_{E,D} = \int \hat{A} \wedge \left(\mathrm{ch}(E^+) - \mathrm{ch}(E^-)\right).$$

Theorem 10.10 (X. Wang & W. Zhang)**.** *Let W be a compact, enlargeable spin manifold and M_1 be any spin manifold of equal dimension. Then $M_1 \# W$ does not have a metric with positive scalar curvature. In particular, $M_1 \# T^n$ does not have PSC.*

Corollary 10.11. *Positive Mass Theorem holds for manifolds of type $AF + X$.*

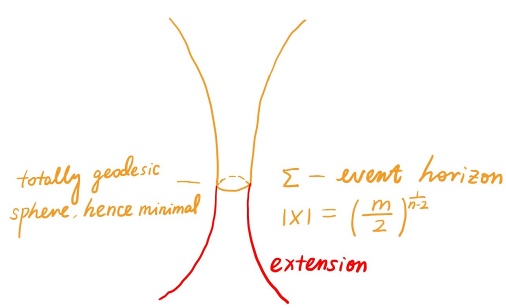
Remark. It was showed by Xianzhe Dai and Junrong Yan that there is also a corresponding index formula for a Witten-type deformation.

11 Riemannian Penrose Inequality

Recall that the Positive Mass Theorem states that for an AF manifold M, the existence of a nonnegative scalar curvature will imply the nonnegative ADM mass. And $m_{ADM} = 0$ if and only if $M = \mathbb{R}^n$. But what if the spacetime has "black holes"?

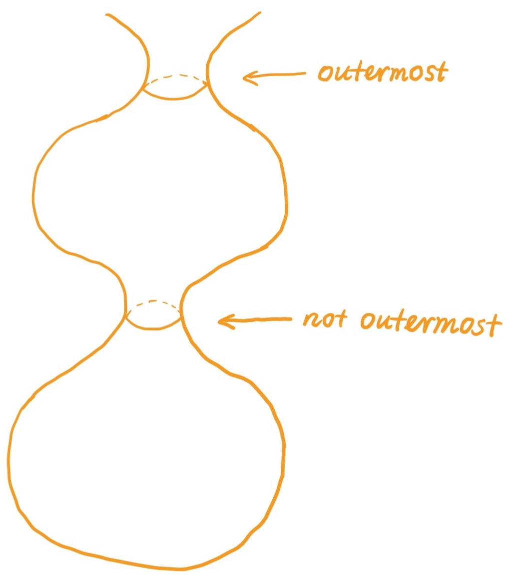
In the theory of general relativity, every black hole is hidden under a so-called event horizon. But from the point of view of mathematics, there are naked singularities, i.e. black holes, that are not hidden under any event horizon. In order to rule out this possibility, Roger Penrose, a famous mathematical physicist and Nobel Laureate in Physics, conceived the **Cosmic Censorship Hypothesis**. And as a test for the hypothesis, Penrose proposed that the total mass of spacetime containing black holes with event horizon are A should be at least $\sqrt{\frac{A}{16\pi}}$ (when $n = 3$).

Remark. It was pointed out by Hawking that A is precisely the entropy of the black hole.



Example 11.1. Schwarzschild Space (model case): $M = \mathbb{R}^n - \{0\}$, $g_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$, $m \geq 0$. In this case $S \equiv 0$, and $m_{ADM} = m$.

Geometrically, these horizons correspond to the outermost minimal hypersurfaces.



Conjecture (Riemannian Penrose Inequality) Let (M^n, g) be an asymptotic flat manifold with $S \geq 0$, and assume Σ is an outermost minimal surfaces. Then

$$m_{ADM} \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Moreover, if " = " holds, then the part M which is outside Σ is isometric to the Schwarzschild space.

The conjecture is proved in the following cases:

$n = 3$ G.Huisken and T.Illmanen proved the case in weaker version, for connected Σ ("The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality", 2001). In general three-dimensional case, H.Bray provided the proof in the paper "Proof of the Riemannian Penrose inequality using the positive mass theorem," 2001.

$n \geq 8$ In 2007, H.Bray and D. Lee verified the conjecture in dimensions less than 8, in the paper "On the Riemannian Penrose inequality in dimensions less than 8."

Proof of Huisken and Ilmanen is based on the heuristic arguments of Geroch and Jang. Namely, Geroch introduced the inverse mean curvature flow and proved that Hawking mass of any connected surface S is monotonic nondecreasing under it.

- Recall the definition of the Hawking mass of any connected surface S

$$m_H(S) = (1 - \int_S H^2) \cdot \sqrt{\frac{|S|}{16\pi}}.$$

Remark. In general, it is a difficult problem in General Relativity to define the mass-energy of a finite region.

In our case, we have:

- $m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}$;
- For $S = \{|x| = R\}$, we have that $m_H(S_R) \longrightarrow m_{ADM}$, as $R \to +\infty$.

Can we construct inverse mean curvature flow interpolating from Σ to S_R? It may be that smooth flow does not exist, but a weak formulation can be made to work!

Now, we will turn to Bray's proof, which is based on the conformal flow. The idea is to construct pairs (M^3, g_t), $0 \le t < +\infty$ such that

- $g_0 = g$ (initial metrics);
- $\forall t \ge 0$ $|\Sigma_t| = |\Sigma|$, (total mass stays invariant);
- $m_{ADM}(g_t)$ is a decreasing sequence (using Positive Mass Theorem);
- $g_t \longrightarrow$ Schwarzschild metric, as $t \to +\infty$.

Under this conditions, it is granted that:

$$m_{ADM}(g) \ge m_{ADM}(g_\infty) = m_{ADM}(Schwarzschild) = \sqrt{\frac{|\Sigma_\infty|}{16\pi}} = \sqrt{\frac{|\Sigma_\infty|}{16\pi}},$$

and Bray's proof is done. The main difficulty is to construct flow g_t satisfying these requirements. In fact, it is possible to find a flow of the previous form

$$g_t = (u(x))^4 g, \quad u_0 = 1.$$

Let g_t be a metric and Σ_t be an assigned "outermost minimal area enclosure". First, we will solve the problem

- $\Delta_g v_t = 0$, outside Σ_t,
- $v_t|_{\Sigma_t} = 0$,
- $\lim_{x\to+\infty} v_t = -e^{-t}$.

Solution of the above PDE problem, we can use to define our required function in the following way:

$$u_t = 1 + \int_0^t v_s ds,$$

and the sketch of the Bray's proof is done.

What about manifolds with spin structure? Witten's Spin Method? Not quite successfull yet!

Herzlich proved the Penrose type inequality in this setting. For start, we will introduce the Yamabe invariant of Σ (or any Riemannian manifold)

$$Y(\Sigma) = \inf_{h \text{ conformal to } g|_\Sigma} \frac{\int_\Sigma S(h) dvol(h)}{(Vol_h(\Sigma))^{\frac{n-2}{n-1}}}$$

Remark. (Yamabe problem) Find a metric in a given conformal class with constant scalar curvature.

R.Schoen solved the remaining cases of Yamabe problem using Positive Mass Theorem. Also, denote

$$S(\Sigma, M) := |\Sigma|^{\frac{1}{n-1}} \inf_{f\in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{\int_\Sigma f^2}$$

Theorem 11.2. *(M. Herzlich) Let (M, g) be asymptotically flat ($\tau > \frac{n-2}{2}$, $n \geq 3$) spin manifold with boundary Σ.. If scalar curvature is nonnegative ($s \geq 0$), and Σ is minimal, and $Y(\Sigma) > 0$, then*

$$m_{ADM} \geq \sigma |\Sigma|^{\frac{n-2}{n-1}},$$

where

$$\sigma = \frac{n-1}{4\pi(n-2)} \frac{S(\Sigma, M)}{1 + \frac{2S(\Sigma,M)}{(\frac{n-2}{n-1}Y(\Sigma))^{\frac{1}{2}}}}.$$

Moreover, equality implies that manifold is isometric to Schwarzschild space.

In fact, the first step of the proof is to use the Positive Mass Theorem for asymptotically flat manifold with boundary.

Theorem 11.3. *(M.Herzlich) Let (M, g) be asymptotically flat spin manifold with boundary Σ. If it has nonnegative scalar curvature, $Y(\Sigma) > 0$ and the mean curvature H of Σ satisfies*

$$H \leq |\Sigma|^{-\frac{1}{n-1}} (\frac{n-1}{n-2} Y(\Sigma))^{\frac{1}{2}},$$

then $m_{ADM} \geq 0$. Moreover, equality implies that M is isometric to $\mathbb{R}^n - B_R(0)$.

BLW:

$$D^2 = \nabla^*\nabla + \frac{s}{4},$$
$$0 = D^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi,$$
$$0 = \int_{r\leq R} < \nabla^*\nabla\psi, \psi > + \frac{s}{4}\psi,$$

and after integrating by parts, we get that

$$0 = \int_{r\leq R} < \nabla\psi, \nabla\psi > + \frac{s}{4} < \psi, \psi >=$$
$$= \int_{r=R} < \nabla_\nu\psi, \psi > - \int_\Sigma < \nabla_\nu\psi, \psi >,$$

where

$$\int_{r=R} < \nabla_\nu\psi, \psi > \longrightarrow |\psi_0|^2 \omega_{n-1} m_{ADM} = \omega_{n-1} m_{ADM},$$

as $R \longrightarrow \infty$.

How do we deal with $\int_\Sigma < \nabla_\nu\psi, \psi >$?

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[1] Friedrich Hirzebruch and Michael Francis Atiyah. Analytic cycles on complex manifolds. *Topology*, 1:25–46, 1962.