Index Theorem, Positive Scalar Curvature, and Enlargibility

Junrong Yan

1 Dirac Operators

1.1 Motivation

According to Einstein's (special) relativity, a free particle of mass m in $\mathbb{R}^3$ with momentum vector $p = (p_1, p_2, p_3)$ has energy

$$E = c\sqrt{m^2c^2 + p^2} = c\sqrt{m^2c^2 + p_1^2 + p_2^2 + p_3^2}.$$

For simplicity, we assume that $c = 1$. Passing to quantum mechanics, one replaces E by the operator $i\frac{\partial}{\partial t}$, and p_j by $-i\frac{\partial}{\partial x_j}$. Therefore the particle now is described by a state function $\Psi(t, x)$ satisfying the equation

$$i\frac{\partial \Psi}{\partial t} = \sqrt{m^2 + \Delta}\Psi.$$

Here the Laplacian

$$\Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}.$$

This motivates Dirac to look for a (Lorentz invariant) square root of Δ. In other words, Dirac looks for a first order differential operator with constant coefficients

$$D = \gamma_j \frac{\partial}{\partial x_j} + m\gamma_0$$

such that $D^2 = m^2 + \Delta$. It follows that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \quad \text{if } 0 \leq i \neq j \leq 3; \quad \gamma_0^2 = 1 \text{ and } \gamma_i^2 = -1 \quad \text{for } i = 1, 2, 3.$$

Dirac realized that, to have solutions, the coefficients γ_i will have to be complex matrices.

1.2 Clifford Algebra

To generalized Dirac operator on higher dimensional manifolds, we introduce Clifford algebra.

Definition 1 (Clliford algebra)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional Euclidean space with an orthonormal basis $\{e_i\}_{i=1}^n$. The Clifford algebra $Cl(V)$ (or denoted by Cl_n) is the real algebra generated by $1, e_1, \cdots, e_n$ subject only to the relations*

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

It is clear that

$$1, e_1, \cdots, e_{n_1} e_1 e_2, \cdots, e_{i_1} e_{i_2} \cdots e_{i_k} \left(i_1 < i_2 < \cdots < i_k\right), \cdots, e_1 \cdots e_n$$

is a vector space basis for Cl_n. Hence $Cl_n \cong \Lambda^* V$ as vector spaces (they are actually isomorphic as Clifford module).

Example 1. *One can see esaily that $Cl_1 \equiv \mathbb{C}$, where e_1 corresponds to i.* $\mathrm{Cl}_2 \equiv \mathrm{H}$*, the quaternions, and the basis vectors $e_1, e_2, e_1 e_2$ correspond to I, J, K.*

Definition 2 (Complexification of Clifford Algebra)**.** *We consider the complexification of the Clifford algebra*

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}.$$

Example 2. *First, one can see essily that*

$$\begin{aligned} \mathbb{C}l_1 &= Cl_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{C} \oplus \mathbb{C}, \\ \mathbb{C}l_2 &= Cl_2 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \equiv End(\mathbb{C}^2). \end{aligned}$$

In fact, one has

Theorem 1. *One has the mod 2 periodicity*

$$\mathbb{C}l_n = \begin{cases} \mathrm{End}\left(\mathbb{C}^{2^{n/2}}\right) & \text{if } n \text{ is even;} \\ \mathrm{End}\left(\mathbb{C}^{2^{(n-1)/2}}\right) \oplus \mathrm{End}\left(\mathbb{C}^{2^{(n-1)/2}}\right) & \text{if } n \text{ is odd}. \end{cases}$$

Definition 3. *A Clifford module (M, c) consists of a $\mathbb{C}$-vector space M and a morphism $c : \mathbb{C}l_n \to End(M)$. Then $\mathbb{C}l_n$ acts on M as matrix multiplication via c.*

Example 3. *The exterior algebra $\Lambda^* V \otimes_{\mathbb{R}} \mathbb{C}$ is a Clifford module, the Clifford action is given by*

$$c(e_i) w = e_i \wedge w - \iota_{e_i} w,$$

where ι is the interior product.

By the mod 2 periodicity, one can see that, when n is even, $\mathbb{C}l_n$ has a canonical $2^{n/2}$-dimensional module, denoted by (Δ_n, c), whose Clifford action c is given by the matrix multiplication; when n is odd, $\mathbb{C}l_n$ has two canonical $2^{n/2}$-dimensional module, denoted by $(\Delta_n^i, c), i = 0, 1$, whose Clifford action c is given by the matrix multiplication of i-th components.

1.3 Dirac operator on $\mathbb{R}^n$

Now, we are in a position to talk about Dirac operator on $\mathbb{R}^n$. Given a Clifford module (M, c), the Dirac operator $D := \sum_{i=1}^n c(e_i)\partial_i$ is a first order differential on M-valued function on $\mathbb{R}^n$. Moreover, one can check easily that $D^2 = \Delta$.

1.4 Dirac operators

Let (X^n, g) be a closed Riemannian manifold of dimension n, locally, for any Clifford module M, the construction in Section 1.3 could be done. The problem is that one can't glue the locally construction usually, and there are some topological obstruction. However, if (X, g) is spin, such construction could be done.

Definition 4. *We say a Riemannian manifold (X, g) is spin if $w_0(X)$ and $w_1(X)$ vanish, where w_0 and w_1 are Stiefel-Whitney of tangent bundle.*

Remark 1. *$w_0 = 0$ iff M is orientable.*

If (X, g) is spin, then

Theorem 2. *There exists a Hermitian vector bundle $(S \to X, \langle\cdot,\cdot\rangle)$, called spinor bundle, such that*

1. *S has a unitary connection ∇^S.*
2. *together with a Clifford action $c : \Gamma(T^*X) \times \Gamma(S) \to \Gamma(S)$ satisfying*
   - *(Leibniz's rule)$\nabla^S(c(v)s) = c(\nabla^{LC}v)s + c(v)\nabla^S s$ for all $v \in \Gamma(T^*M), s \in \Gamma(S)$, where ∇^{LC} is the Levi-Civita connection.*
   - *If $g(v, v) = 1$, then $\langle c(v)s_1, c(v)s_2\rangle = \langle s_1, s_2\rangle$ for all $v \in \Gamma(T^*M), s_1, s_2 \in \Gamma(S)$.*

Moreover, suppose locally $\nabla^{LC}e_i = \sum_j w_{ij}e_j$, then connection ∇^S could be given by

$$\nabla^S = d + \sum_{i,j} \frac{w_{ij}}{4} c(e_i)c(e_j). \tag{1}$$

Remark 2. *When $M = \mathbb{R}^n$, $S := \mathbb{R}^n \times \Delta_n$.*

Example 4.

- *$T^n, \mathbb{R}^n$, any Lie group G and any 3 dimensional orientable manifolds are spin, since their tangent bundle are trivial.*
- *All orientable surfaces are spin.*
- *A complex manifold X is spin iff $c_1(X) \equiv 0 (\mathrm{mod} 2)$.*
- *$\mathbb{RP}^n$ is spin iff $n \equiv 3 \bmod 4$; $\mathbb{CP}^n$ is spin iff n odd ($n \equiv 1 \bmod 2$); HP^n is always spin.*
- *Since $\{w_i\}$ are homotopy invariants, hence if X and Y are homotopic equivalent, then X is spin iff Y is spin.*

1.5 Lichnerowicz Formula

Definition 5. *If (X, g) is spin, and let $S \to X$ be the spinor bundle, then the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is defined by*

$$D := \sum_i c(e_i) \nabla^S_{e_i},$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame of T^*X.

Theorem 3. $D^2 = \Delta + \frac{k}{4}$, *where k is the scalar curvature on X.*

Proof. Assume that at $p \in X$, $\nabla^{LC} e_i = 0$, then by a straightforward computation,

$$\begin{aligned}
D^2 :&= \sum_{i,j} c(e_i) \nabla^S_{e_i} c(e_j) \nabla^S_{e_j} \\
&= \sum_i c(e_i) \nabla^S_{e_i} c(e_i) \nabla^S_{e_i} + \sum_{i \neq j} c(e_i) \nabla^S_{e_i} c(e_j) \nabla^S_{e_j} \\
&= \sum_i c(e_i) c(e_i) \nabla^S_{e_i} \nabla^S_{e_i} + \sum_{i \neq j} c(e_i) c(e_j) \nabla^S_{e_i} \nabla^S_{e_j} \text{ (By Leibniz's rule and } \nabla^{LC} e_i = 0) \\
&= -\sum_i \nabla^S_{e_i} \nabla^S_{e_i} + \sum_{i<j} c(e_i) c(e_j) (\nabla^S_{e_i} \nabla^S_{e_j} - \nabla^S_{e_j} \nabla^S_{e_i}) \text{ (Since } c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij}) \\
&= \Delta + \sum_{i<j} c(e_i) c(e_j) R^S(e_i, e_j) \\
&= \Delta + \frac{1}{8} R_{ijkl} c(e_i) c(e_j) c(e_k) c(e_l) \text{ (By (1))} \\
&= \Delta + \frac{1}{8} \sum_l \left[\frac{1}{3} \sum_{i,j,k} (R_{ijkl} + R_{jkil} + R_{kijl}) c(e_i) c(e_j) c(e_k) \right. \\
&\left. + \sum_{i,j} R_{ijil} c(e_i) c(e_j) c(e_i) + \sum_{i,j} R_{ijjl} c(e_i) c(e_j) c(e_j) \right] c(e_l) \\
&= \Delta + \frac{1}{4} R_{ijil} c(e_j) c(e_l) \text{ (By Bianchi identity)} \\
&= \Delta - \frac{1}{4} \operatorname{Ric}(e_j, e_l) c(e_j) c(e_l) \\
&= \Delta + \frac{1}{4} \operatorname{Ric}(e_j, e_l) \delta_{jl} \\
&= \Delta + \frac{k}{4}
\end{aligned}$$

∎

When (X, g) admits PSC, D^2 is a strictly positive operator, hence by Atiyah-Singer index theorem

Theorem 4. *Let (X, g) be closed and spin, then if (X, g) admits PSC, then $\hat{A}$-genus vanishes.*

See next section for the definition of hatA-genus.

Remark 3. *The inverse is not ture. In fact, by a more refined argument (we introduce the notion of enlargibility), one can show that T^n can't admit a metric of PSC, but its $\hat{A}$-genus vanishes. Indeed, one can prove that if X is closed and spin, $X\#T^n$ cannot admit a metric of PSC (we prove this before in the lower dimension using the minimal surface technique without assuming the spin condition).*

2 Enlargeability

Last time: Let (M, g) AF with nonnegative scalar curvature. Then $M_1\#T^n$ has no PSC implies that the PMT holds on M.

We proved earlier in the term that if $3 \leq n \leq 7$ and M^n is closed then $M^n\#T^n$ has no PSC. So for $3 \leq n \leq 7$ we get PMT without needing M to be spin. Our current goal is to show that if M is closed and spin then $M^n\#T^n$ has no PSC, which gives a proof of PMT for the spin case.

Definition 6. *Suppose $f : X^n \to Y^n$ is C^1 and $\epsilon > 0$. We say f is ϵ-*contractible *if for all $p \in X$, $f_* : T_pY \to T_{f(p)}Y$ is ϵ-contractible; that is, for all $v \in T_pX$, $\|f_*v\|_Y \leq \epsilon\|v\|_X$.*

Example 5. *For any $\epsilon > 0$ there is an ϵ-contractible map $f; \mathbb{R}^n \to S^n(1)$.*

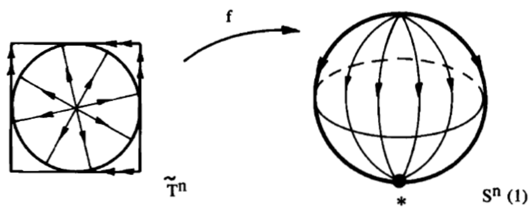
Definition 7. *A compact riemannian n-manifold is said to be enlargeable if for every $\varepsilon > 0$ there exists an orientable riemannian covering space which admits an ε-contracting map onto $S^n(1)$ which is constant at infinity and of non-zero degree. If for each $\varepsilon > 0$, there is a finite covering space with these properties, we call the manifold compactly enlargeable.*

Remark 1. *A map is constant at infinity if it is constant outside a compact set. The degree of such a map $f : X \to S^n$ is defined as*

$$\deg(f) = \frac{\int_X f^*\omega}{\int_{S^n} \omega}$$

where ω is an n-form on S^n with non-zero integral. The degree can also be defined as usual in terms of signed counting of pre-images of of f at regular values.

The square flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is certainly enlargeable since the universal covering space has the required mappings for all $\varepsilon > 0$. This torus is, in fact, compactly enlargeable. We see this as follows. For each $k \in \mathbb{Z}^+$, the lattice $(k \cdot \mathbb{Z})^n \subset \mathbb{Z}^n$ gives a k^n-fold covering torus $\tilde{T}^n \equiv \mathbb{R}^n/(k \cdot \mathbb{Z})^n$, which admits the (π/k)-contracting map to $S^n(1)$ of degree 1 pictured above.



Theorem 1. *The following statements hold in the category of compact manifolds:*

(A) Enlargeability is independent of the riemannian metric.

(B) Enlargeability depends only on the homotopy-type of the manifold.

(C) The product of enlargeable manifolds is enlargeable.

(D) The connected sum of any manifold with an enlargeable manifold is again enlargeable.

(E) Any manifold which admits a map of non-zero degree onto an enlargeable manifold is itself enlargeable.

Proof. It is evident that $(E) \Rightarrow (B) \Rightarrow (A)$ and that $(E) \Rightarrow (D)$. To prove (E) we consider two compact oriented riemannian n-manifolds X and Y, and a map $F : X \to Y$ of non-zero degree. By compactness there exists a $c > 0$ so that $\|dF\| \leqq c$ on X (i.e., F is c-contracting). Given $\varepsilon > 0$, there is a riemannian covering space $p : \tilde{Y} \to Y$ which admits a (ε/c)-contracting map $f : \tilde{Y} \to S^n(1)$ which is constant outside a compact set $\tilde{K} \subset \tilde{Y}$ and of nonzero degree. Taking the fibre product of p and F gives a covering space $p' : \tilde{X} \to X$ and a proper mapping $\tilde{F} : \tilde{X} \to \tilde{Y}$ so that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{F}} & \tilde{Y} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{F} & Y \end{array}$$

commutes. Since $\tilde{F}$ is a lifting of F, we have $\|\nabla \tilde{F}\| \leqq c$ on $\tilde{X}$. Hence, the composition $f \circ \tilde{F} : \tilde{X} \to S^n(1)$ is ε-contracting. Since $\tilde{F}$ is proper, we see that $f \circ \tilde{F}$ is constant outside the compact set $\tilde{F}^{-1}(\tilde{K})$. It is easy to see that: $\deg(f \circ \tilde{F}) = \deg(f)\deg(F) \neq 0$. Hence, X is enlargeable as claimed.

To prove (C), we fix a degree-1 map $\phi : S^n(1) \times S^m(1) \to S^{n+m}(1)$ (Recall that $S^{n+m} \cong S^n \times S^m / S^m \vee S^n$) and let $c = \sup \|d\phi\|$. This map is chosen to be constant on the set $(S^n(1) \times \{*\}) \cup (\{*\} \times S^m(1))$, where each "$*$" denotes a distinguished point in the sphere. Suppose now that we are given (ε/c)-contracting maps, $f : X^n \to S^n(1)$ and $g : X^m \to S^m(1)$, which are constant $(= *)$ at infinity and of non-zero degree. Then the map $\phi \circ (f \times g) : X^n \times Y^m \longrightarrow S^{n+m}(1)$ is ε-contracting, constant at infinity and of non-zero degree. From here the argument is straightforward. $\square$

Theorem 2. *An enlargeable spin manifold X cannot carry a metric of positive scalar curvature.*

2.1 Review on index Theorem and Lichnerowicz Formula

Theorem 3. *Let M be a closed Spin manifold, $S \to M$ be the spinor bundle with spinor connection ∇^S. Let $E \to M$ be a complex vector bundle with a unitary connection ∇^E. On $S \otimes E$, one has connection $\nabla^{S\otimes E} := \nabla^S \otimes 1 + 1 \otimes \nabla^E$, i.e., for any $s \in \Gamma(s), e \in \Gamma(E)$, $\nabla^{S\otimes E} s \otimes e = \nabla^S s \otimes e + s \otimes \nabla^E e$. Also, $S \otimes E$ admits a clifford acction, such that for any $X \in \Gamma(TM)$, $c(X)s \otimes e = (c(X)s) \otimes e$. Let $D^{S\otimes E} := \sum_i c(e_i)\nabla^{S\otimes E}_{e_i}$ be the Dirac operator (where $\{e_i\}$ is a local orthonormal frame), then*

$$ind(D^{S\otimes E}) = \int_M \hat{A}(M)ch(E).$$

Theorem 4 (Lichnerowicz formula)**.**

$$(D^{S\otimes E})^2 = \Delta + k/4 + \mathcal{R}^E,$$

where k is the scalar curvature of M, Δ is the connection Laplacian with respect to $\nabla^{S\otimes E}$, $\mathcal{R}^E := \sum_{i,j} c(e_i)c(e_j)R^E(e_i, e_j)$, R^E is the curvature on E.

2.2 Quick introduction to Chern-Weil theory

Let $E \to M$ be a smooth complex vector bundle over a smooth compact manifold M. We denote by $\Omega^*(M; E)$ the space of smooth sections of the tensor product vector bundle $\Lambda^*(T^*M) \otimes E$ obtained from $\Lambda^*(T^*M)$ and E :

$$\Omega^*(M; E) := \Gamma\left(\Lambda^*(T^*M) \otimes E\right).$$

Definition 1. *A connection ∇^E on E is a $\mathbb{C}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M; E)$ such that for any $f \in C^\infty(M), X \in \Gamma(E)$, the following Leibniz rule holds,*

$$\nabla^E(fX) = (df)X + f\nabla^E X.$$

Just like the exterior differential operator d, a connection ∇^E can be extended canonically to a map, which we still denote by ∇^E,

$$\nabla^E : \Omega^*(M; E) \longrightarrow \Omega^{*+1}(M; E)$$

such that for any $\omega \in \Omega^*(M), X \in \Gamma(E)$,

$$\nabla^E : \omega X \mapsto (d\omega)X + (-1)^{\deg \omega}\omega \wedge \nabla^E X$$

Definition 2. *The curvature R^E of a connection ∇^E is defined by*

$$R^E = \nabla^E \circ \nabla^E : \Gamma(E) \to \Omega^2(M; E),$$

which, for brevity, we will write $R^E = \left(\nabla^E\right)^2$.

One can see that R^E may be thought of as an element of $\Gamma(\mathrm{End}(E))$ with coefficients in $\Omega^2(M)$. In other words,

$$R^E \in \Omega^2(M; \mathrm{End}(E))$$

To give a more precise formula, if $X, Y \in \Gamma(TM)$ are two smooth sections of TM, then $R^E(X,Y)$ is an element in $\Gamma(\mathrm{End}(E))$ given by

$$R^E(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}.$$

Finally, in view of the composition of the endomorphisms, one sees that for any integer $k \geq 0$,

$$\left(R^E\right)^k = \overbrace{R^E \cdot \cdots \cdot R^E}^{k} : \Gamma(E) \longrightarrow \Omega^{2k}(M; E)$$

is a well-defined element lying in $\Omega^{2k}(M; \mathrm{End}(E))$.

For any smooth section A of the bundle of endomorphisms, End (E), the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map

$$\mathrm{tr} : \Omega^*(M; \mathrm{End}(E)) \longrightarrow \Omega^*(M)$$

such that for any $\omega \in \Omega^*(M)$ and $A \in \Gamma(\mathrm{End}(E))$,

$$\mathrm{tr} : \omega A \mapsto \omega\, \mathrm{tr}[A].$$

We still call it the function of trace.

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

be a power series in one variable. Let R^E be the curvature of a connection ∇^E on E. The trace of

$$f\left(R^E\right) = a_0 + a_1 R^E + \cdots + a_n \left(R^E\right)^n + \cdots$$

is an element in $\Omega^*(M)$. We can now state a form of the Chern-Weil theorem as follows.

Theorem 5. *(i) The form $\mathrm{tr}\left[f\left(R^E\right)\right]$ is closed. That is,*

$$d\, \mathrm{tr}\left[f\left(R^E\right)\right] = 0.$$

(ii) If $\widetilde{\nabla}^E$ is another connection on E and $\widetilde{R}^E$ its curvature, then there is a differential form $\omega \in \Omega^*(M)$ such that

$$\mathrm{tr}\left[f\left(R^E\right)\right] - \mathrm{tr}\left[f\left(\widetilde{R}^E\right)\right] = d\omega.$$

Since

$$\det\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)=\exp\left(\operatorname{tr}\left[\log\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$$

in view of the following power series expansion formulas for $\log(1+x)$ and $\exp(x)$

$$\log(1+x)=x-\frac{x^2}{2}+\cdots+\frac{(-1)^{n+1}x^n}{n}+\cdots$$

and

$$\exp(x)=1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\cdots.$$

By Theorem 5, the Chern class

$$c(E)=\left[\det\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\in H^*(M,\mathbb{C})$$

is some summation of even cohomologies, i.e., one has

$$c(E)=1+c_1(E)+\cdots+c_k(E)+\cdots$$

with each i-th Chern class

$$c_i(E)\in H^{2i}(M).$$

(Here for a closed differential form w, $[w]$ denotes the cohomology represented by w.)

Similarly, the Chern character and $\hat{A}$-class are defined by

$$\operatorname{ch}(E,)=\left[\operatorname{tr}\left(\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right)\right]\in H^{\text{even}}(M),$$

$$\widehat{A}(E)=\left[\det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right)^{1/2}\right)\right].$$

It follows from the definition that $\hat{A}_0(E)=1$.

Moreover, $\hat{A}(M):=\hat{A}(TM\otimes\mathbb{C})$.

Definition 3. *For $[w]\in H^*(M)$, we define the pairing*

$$\langle[w],[M]\rangle:=(\int_M w=)\int_M w_n,$$

where w_n is the top degree components of w.

2.3 Proof of Theorem 2

For clarity's sake we only present here a proof for the case of compactly enlargeable manifolds.

Let X be a compactly enlargeable n-manifold, and suppose X carries a metric with $\kappa \geqq \kappa_0$ for a constant $\kappa_0 > 0$. We may assume that X has even dimension $2n$. (If not, replace X by $X \times S^1$.)

Choose a complex vector bundle E_0 over the sphere $S^{2n}(1)$ with the property that the top Chern class $c_n(E_0) \neq 0$. (This is certainly possible, cf. [?]). We now fix a unitary connection ∇^{E_0} on E_0 and we let R^{E_0} denote the curvature 2-form. Moreover,

$$\mathrm{ch}(E_0) = rank(E_0) + \frac{1}{(n-1)!} c_n(E_0).$$

Let $\varepsilon > 0$ be given and choose a finite orientable covering $\tilde{X} \to X$ which admits an ε-contracting map $f : \tilde{X} \to S^{2n}(1)$ of non-zero degree. Using f, we pull back the bundle E_0, with its connection, to $\tilde{X}$. This gives us a bundle $E \equiv f^* E_0$ with connection $\nabla^E \equiv f^* \nabla^{E_0}$. We then consider the complex spinor bundle S of $\tilde{X}$ with its canonical riemannian connection, and consider the Dirac operator $D^{S\otimes E}$ on the tensor product $S \otimes E$. We know from Theorem 4 that

$$(D^{S\otimes E})^2 = \Delta + \frac{k}{4} + \mathcal{R}^E$$

where $\mathcal{R}^E$ depends universally and linearly on the components of the curvature tensor R^E of E, Moreover

$$\left\|\mathcal{R}^E\right\| \leq C|f_*|^2|R^{E_0}| \leq C'\varepsilon^2$$

for some $C > 0$.

Hence if ε is small, by Theorem 4, $D^{S\otimes E}$ is invertible, hence $ind(D^{S\otimes E}) = 0$.

However, let $m = rank(E_0)$ this index is given by

$$\begin{aligned}
\mathrm{ind}\left(D^{S\otimes E}\right) &= \langle \mathrm{ch}\, E \cdot \hat{A}(\tilde{X}), [\tilde{X}] \rangle \\
&= \int_{\tilde{X}} \left(m + \frac{1}{(n-1)!} c_n(E) \right) \cdot \hat{A}(\tilde{X}) \\
&= m \int_{\tilde{X}} \hat{A}_n(\tilde{X}) + \int_{\tilde{X}} \frac{1}{(n-1)!} c_n(E) \hat{A}_0 \\
&= \int_{\tilde{X}} \frac{1}{(n-1)!} c_n\left(f^* E_0\right) \text{ (By Theorem 4)} \\
&= \int_{\tilde{X}} \frac{1}{(n-1)!} f^*\left(c_n\left(E_0\right)\right) \\
&= \frac{1}{(n-1)!} \deg(f) \int_{S^{2n}} c_n\left(E_0\right) \\
&\neq 0,
\end{aligned}$$

which is a contradiction.